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Research Trends in Mathematical and Statistical Science

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PREFACE

*We are delighted to publish our book entitled **"Research Trends in Mathematical and Statistical Science"**. This book is the compilation of esteemed articles of acknowledged experts in the fields of basic and applied mathematical science.*

This book is published in the hopes of sharing the excitement found in the study of mathematics and statistical science. Mathematical science can help us unlock the mysteries of our universe, but beyond that, conquering it can be personally satisfying. We developed this digital book with the goal of helping people achieve that feeling of accomplishment.

The articles in the book have been contributed by eminent scientists, academicians. Our special thanks and appreciation goes to experts and research workers whose contributions have enriched this book. We thank our publisher Bhumi Publishing, India for taking pains in bringing out the book.

Finally, we will always remain a debtor to all our well-wishers for their blessings, without which this book would not have come into existence.

- Editorial Team

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CONTENTS

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AN INTRODUCTION TO SOME NATURE BASED OPTIMIZATION TECHNIQUES

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Abstract:

The process, by which an optimal solution is chosen among many alternative solutions, is called optimization. For several issues in real life, it is not feasible to verify all solutions within a reasonable time. Algorithms inspired by nature are stochastic techniques which are designed to deal with such issues. They normally incorporate some deterministic and haphazardness methods together and afterward iteratively think about a number of arrangements until an agreeable one is found. Many optimization techniques were developed to tackle different linear, non linear and multi objective problems. While in recent years different optimization approaches have been developed, some common optimization techniques presented here are Ant colony optimization, Honey Bee Algorithm, Cat Swarm optimization algorithm and Cuckoo optimization algorithm.

Introduction:

Optimization is the analysis of dynamic preparation, constructing, and resolving day to day life issues and problems. These issues can be related to each sector like industrial sector, scientific sector, knowledge sector etc. In optimization, the primary goal is to optimize the physical parameter in considered problem. Many organizations use these methods to optimize their income with minimum expenditures, travel expenses, maximize power, minimize flaws, etc. Nature based algorithms are extracted by captivating the stirring inspiration of biological species's collective conduct and decentralized management structure. Scientists and researchers have solved real life issues with mathematical or simulation modelling by carefully studying the underlying individual behaviors and observations. Developing heuristics from nature is a continuous process as world is filled by more than 84,00,000 species. Optimization techniques developed during Second World War. Many researchers and scientists are inspired by nature based algorithms and implement different terminologies and advanced operators on natural choice and growth, to attain solidity and convergence in Multiple Search space. They used it to solve various linear, non-linear, uniform and non- uniform complex problems with minimum time consumption. Nature based algorithms were developed on behavior of Bee, Cuckoo, Monkey, wolf spider, elephant, Bat, fire fly, ants etc.

Ant Colony Optimization Algorithm:

Ant colony optimization technique is based on behavior of ants searching for food. It is searching for best path in the graph based on behavior of ants seeking a path between their colony and source of food. It is a Meta-heuristic optimization method. Originally it was proposed by Marco Dorigo in 1992 [1, 2]. Ants navigate from nest to food source. Ants discover shortest track via pheromone trails as they are blind by nature. Each ant moves at arbitrarytrack. Pheromone is dropped on track. More pheromone on track rises probability of path being tracked. One of the chiefconcepts behind this method is that the ants can link with one another through subsidiary means by making alterations to the attention of highly unstable chemicals called pheromones in their instant environment. The behavior of ants for finding shortest track is shown by Fig.1.

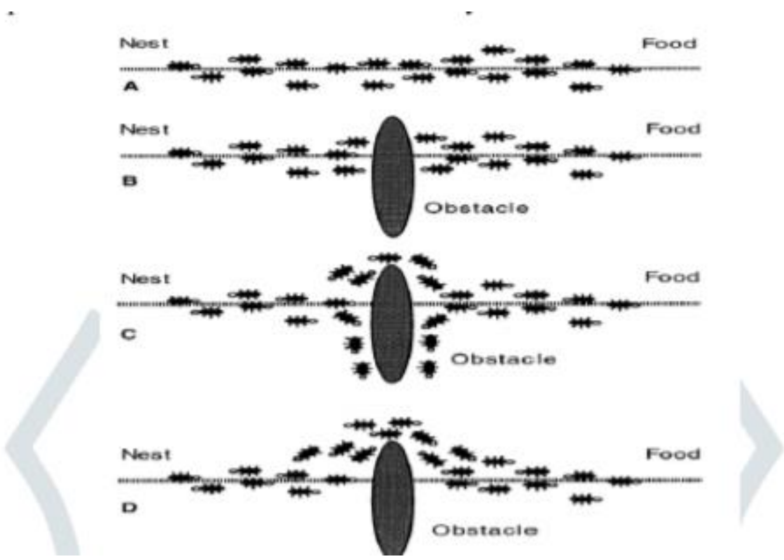

Figure 1: Behavior of Ants to find shortest route [2, 3, 4]

Honey Bee Optimization Algorithm:

The bee's algorithm is a population-based search algorithm. It was first developed by Pham, Ghanbarzadeh *et al.* [5]. This algorithm stimulated by the usual searching performance of honey bees. Honeybees are social bugs that live in groups composed of a single queen and up to numerous thousand workers.Their collective behavior enables it to solve many complex tasks such as keeping a persistenthive temperature, keeping track of changing searching conditions and choosing the finest possible nest spot. In searching of nest site, some queen cells are produced to generate new queen. Before its birth, the old queen leaves the colony with the half of the colony components to form a new colony. They search new nest site. The scouts seek about twelve nest sites. They indicate the various locations of new nests by waggle dances. The dance quality is related to the nest site quality. Thus, and over time, selected sites decrease until a single site will be found. In Food source searching, some bees "scouts" navigate and explore the region in aim to find a food source. In the positive case, they come at the hive in place called "dance floor" to transmit and share this discovery with the others through dance language (round or waggle dance relating to the discovery distance). Some bees are recruited and then, become foragers. Their number is proportional to the food quantity information communicated by the scouts. We call this step exploration phases which is followed by the exploitation step. Bee collects food and calculates their quantity to make a new decision. Either it continues collecting by the memorization of this best location, or it leaves the source and returns to hive as simple bee [6-8].

Cat Swarm Optimization Algorithm:

Cat swarm optimization is a single-objective algorithm. Chu *et al.* [9] in invented this algorithm 2006. It is encouraged by resting and tracing behaviors of cats. Cats are lazy in nature and devotemaximum of time in resting, although their awareness is very high during resting. So, they are regularlydetecting the surroundswisely and consciously.Upon seeing a target, they start stirring towards it quickly. Therefore, CSO algorithm is constructed on combination of these two main manners of cats. CSO algorithm is composed of two modes: tracing and seeking modes. A solution set is represented by each cat having its position flag and fitness value. The position is prepared with dimensions in the search space, and every dimension has its individual speed; the fitness value describe wellness of the solution set (cat) is; the flag is to categorize the cats into seeking or tracing mode. Thus, we should first identify number of cats engaged in the iteration and run them from first to last in the algorithm. The finest cat in each iteration is kept into memory, and the one at the ending iteration represent the solution. The population of cats depicts more information than just the activities or poses. It helps in investigation and exploitation of search space all together and predictably leads to optimality. CSO can even be quicker in finding the solution, in particular, when the population is well break up and indomitable [10].

Cuckoo Optimization Algorithm:

The optimization algorithm inspired by life of bird Cuckoo is novel evolutionary algorithm for non linear optimization problems. X. S. Yang and S. Deb [11] expanded this algorithm in 2009 and it was investigated in detail by R. Rajabioun [12] in 2011. The main inspiration behind this algorithm is egg laying and reproduction characteristics of cuckoo bird. This algorithm starts with initial population. This population owns eggs which are placed in nests of other birds.

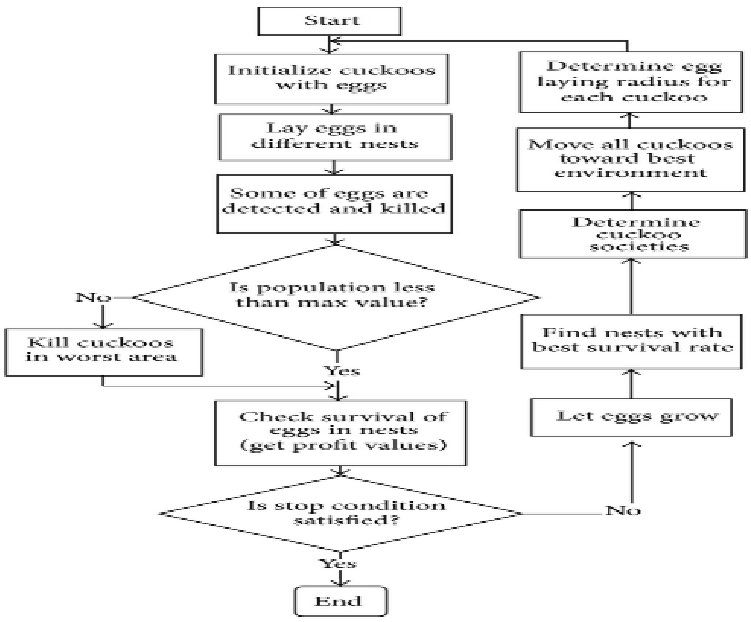

Figure 2: Flow chart of Cuckoo Optimization Algorithm [11]

Cuckoo optimization algorithm is based on survival of these eggs. In reality, this idle bird absolutely enables other birds to play an involuntary part in the survival of its generation. The host bird identifies and destroys the eggs that have fewer similarities with its eggs. Cuckoos develop and learn the consistent way of laying eggs similar to host bird's eggs how to consistently lay eggs much like the eggs of the target host bird, and the host birds learn the ways of recognizing the fake eggs. Cuckoos explore for the most appropriate spot to put down eggs for maximum survival rate of eggs. They build several communities after the remaining eggs develop and turn into an adult cuckoo. Every community has its own habitat for living. The most excellent habitat of all communities will be the target for the cuckoos in other communities. This strategy proceeds until the best position is achieved for the highest benefit value and much of the cuckoo population is accumulated around the same position.

Conclusion:

Many problems in daily life are of complex nature with non linear constraints. These problems can be easily converted in optimization problems. Traditional optimization methods cannot be suitable for these complex nature problems. These problems require refined optimization methods. Nature based algorithms are best choice as they are flexible and capable of solving these type of problems. Nature based algorithms are increasing rapidly because of their various applications in diverse fields.

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RELIABILITY ANALYSIS FOR COMPONENTS USING FUZZY MEMBERSHIP FUNCTIONS

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Introduction to Reliability:

Reliable engineering is one of the important engineering tasks in design and development of a technical system. In the last 30 years, much effort has been made in the design and development of reliable large-scale systems for space science, Military applications, and power distribution.

In the real world problems, the collected data or system parameters are often imprecise, because of incomplete or non-obtainable information. The probabilistic approach to the conventional reliability analysis is inadequate to account for such built-in uncertainties in data. For this reason the concept of fuzzy reliability have been introduced and formulated as a transition from fuzzy success state to fuzzy failure state.The reliability of system is defined as the probability that the system performs its assigned function properly during a predefined period.

Most of the research in classical reliability theory is based on binary state assumption for states. For multi-component system with parallel redundancy, graceful degradation describes a smooth change to lower performance level of the system as individual component fail. For such systems, it is therefore unrealistic to assume that the system possess only two stages, i.e. “Working” or “Failed”. Such systems may be considered working to a certain degree at different states of its performance degradation during its transition from fully working state to completely failed state. The degree may be any real number ‘0’ (to indicate the system is in failed state) and ‘1’ (to indicate the system is in working state). The new concept of fuzzy sets in reliability analysis is suggested in. Fuzzy sets can express the gradual transition of the system from a working state to a failed state. The crisp set theory only discuss the system in to a working state and failed state while fuzzy set theory can handle all possible states between fully working state and completely failed state. This approach to the reliability theory is known as PROFUST reliability theory. In PROFUST reliability theory the binary state assumptions are replaced by fuzzy state assumptions.

The definition of Reliability Engineering:

Immediately following reliability's emergence as a technical discipline just after World War I, it was used to compare operational safety of airplanes. Reliability was then measured as the number of accidents per hour of flight time. Reliability engineering was considered as an equal to applied probability and statistics. Nowadays, reliability research has been clearly sub-divided into smaller entities and research topics may be divided by the methodology that applies: mathematics based approaches have a long history, especially in reliability analysis of large system; while physics based approaches are being introduced especially in component level studies.

Reliability theory is a body of ideas, mathematical models, and methods directed to predict, estimate, understand, and optimize the lifespan distribution of system and their components. The term reliability is defined as the probability that a component or a system will perform a required function for a given period of time when used under stated operating conditions. This definition has its roots in military handbook MIL-STD-721C.

ISO however has a different and more general definition of what the term reliability means. ISO describes reliability as the ability of item to perform a required function, under given environment and operational conditions and for a stated period of time. Others measures include Maintenance Free Operating (MFOP), which allows a period of operation during which an item will be able to carry out all its assigned missions, without the operator being restricted in any way due to system faults of limitations, with the minimum of maintenance.

There are also as many ways to measure reliability as there are different ways to define it. The most widely used measure of reliability are Mean Time to Failure (MTTF) and the Mean Time between Failure (MTBF), which is the mean, or the expected value, of a probability distribution. One key issue regarding reliability measurement/estimation is that no one can calculate the exact period of time for which a component will work without a failure. The only thing that any model can do is to calculate the probability of the component working without failure for a period of time.

Many systems are designed to operate for specified period of time. If there is no benefit to having the system last longer than what the customers need, then it may be a waste of resources to over-design the system's capabilities. But in many situations, this is not the case. For example, the Mars rovers; Sprit and Opportunity, were for 90-day missions and those missions turned into much longer than that. In this case, the Scientific, benefits of this extra system life have been tremendous.

The basic assumptions is the when randomly selecting a large sample from a very large population, the sample will possess the same properties and behavior as the total population. It is important to understand that such a description explains what happens when a large number of components are put into operation. The resulting reliability calculations have no meaning when applied to a single item.

To express this relationship mathematically, the continuous random variable T is defined as the time to failure of a system or a component where $T \geq 0$. Then the reliability can be express as:

$$F(t) = P_r\{T < t\}$$

$$R(t) = 1 - F(t) = P_r\{T \geq t\}$$

Where, $F(t)$ is defined as the probability that a failure occurs before time t. There are several other definitions for reliability estimation in literature, but in general most of the scientists and researchers describe reliability in terms of performance without failure under stated conditions for specified period of time.

In this Chapter, we present a new method to analyze fuzzy system reliability using fuzzy number arithmetic operations. Suggested method is based on PROFUST reliability theory, where the reliability of each system component is presented by trapezoidal fuzzy number. The proposed method uses simplified fuzzy arithmetic operations of fuzzy numbers.

Fuzzy arithmetic operations for reliability analysis:

We briefly review some basic definitions of fuzzy sets from [1]. Let U be the universe of discourse, $U = \{u_1, u_2, u_3, \ldots\ldots, u_n\}$ .Let set $\tilde{R}$ of U is a set of ordered pairs:

$\{(u_1, f_{\tilde{R}}(u_1)), (u_2, f_{\tilde{R}}(u_2)) \ldots\ldots (u_n, f_{\tilde{R}}(u_n))\}$

Where, $f_{\tilde{R}}$, $f_{\tilde{R}} : U \to [0,1]$ is the membership function of U_i in $\tilde{R}$, and $f_{\tilde{R}}(u_i)$ indicates the grade of membership U_i in $\tilde{R}$.

To formulate the fuzzy numbers or parameters, we can use either membership functions or possibility distributions [91]. In this chapter we use the trapezoidal membership function. The membership function curve and characteristic of the trapezoidal fuzzy number $\tilde{R}_i = (\alpha_i, \beta_i, \gamma_i, \delta_i)$, $1 \leq i \leq n$. are expressed in Fig. (1.1.1) and eqn. (1.1.1)

Where α_i and δ_i are called the left and right spreads of the curve, respectively.

$f_{\tilde{R}}(u_i)$

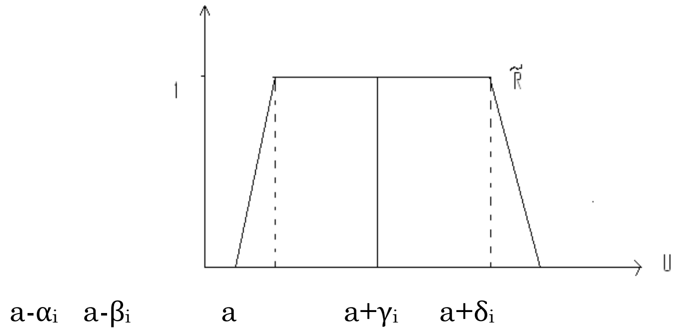

Figure 1.1.1: Trapezoidal membership function

The fuzzy trapezoidal distribution is:

$$\mu_{\tilde{R}}(U) = \begin{cases} 0 & U \le a - \alpha_i \\ \alpha_i + U - a/\alpha_i - \beta_i & a - \alpha_i < U \le a - \beta_i \\ 1 & a - \beta_i \le U < a + \gamma_i, \\ \delta_i - U + a/\delta_i - \gamma_i & a + \gamma_i \le U \le a + \delta_i \\ 0 & a + \delta_i < U \end{cases} \qquad (1.1.1)$$

Where $0 < \beta_i < \gamma_i$ and $a \in \Box$

The methodology for fuzzy system reliability analysis:

In this section, we present the technique for analyzing system reliability using fuzzy arithmetic operations. The serial system and parallel system are shown in fig. (1.1.3) and fig. (1.1.4).

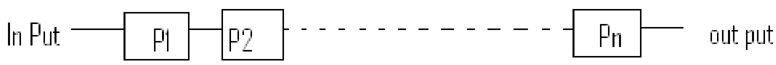

Figure 1.1.3: Configuration of a Serial system

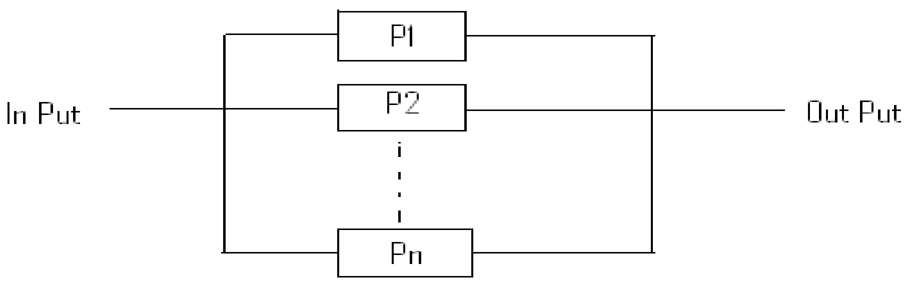

Figure 1.1.4: Configuration of a Parallel system

Where P_i's are subsystems of equipment. The subsystems P_i's are represented by the trapezoidal fuzzy number.

Suppose that the event model contains logical operators "OR" and "AND" then, the reliability function of the system can be obtained replacing the logical operators by the algebraic addition and multiplication. From the Boolean event of the system, a probabilistic model of the structure is obtained with the difference that the logical OR and AND operators. Creating the nodes of the model, are replaced by the appropriate probability operators like fig.5.

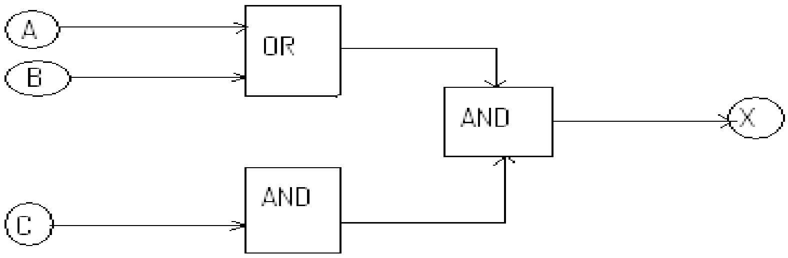

Figure 1.1.5: The probabilistic model of system structure

The probability function of AND and OR operators are:

$$P_X^{AND} = \prod_{i=1}^{n} R_i \tag{1.1.2}$$

$$P_X^{OR} = 1 - (1 - R_1)(1 - R_2).....(1 - R_n) \tag{1.1.3}$$
$$= 1 - \prod_{i=1}^{n}(1 - R_i)$$

Where R_i's are the probabilities of input events, P_x those of out put events

Fuzzy operators of reliability analysis:

The membership function of the fuzzy AND and OR operators can be obtained, considering the variables in equation (1.1.2) and (1.1.3) as fuzzy variables and substituting the algebraic operations given in section 2.

The fuzzy form of AND operator function is:

$$\tilde{P}_X^{AND} = \prod_{i=1}^{n} \tilde{R}_i = AND(\tilde{R}_1, \tilde{R}_2, ..., \tilde{R}_n) \tag{1.1.4}$$

And the fuzzy form of OR operator function is:

$$\tilde{P}_X^{OR} = 1 - \prod_{i=1}^{n}(1 - \tilde{R}_i) = OR(\tilde{R}_1, \tilde{R}_2, ..., \tilde{R}_n) \tag{1.1.5}$$

The reliability of the serial system shown in fig. (5.3) can be evaluated and is equal to:

$$\tilde{R}_1 \otimes \tilde{R}_2 \otimes ..... \otimes \tilde{R}_n = (\alpha_1, \beta_1, \gamma_1, \delta_1) \otimes (\alpha_2, \beta_2, \gamma_2, \delta_2) \otimes ..... \otimes (\alpha_n, \beta_n, \gamma_n, \delta_n) \tag{1.1.6}$$
$$= \left[\prod_{i=1}^{n}(\alpha_i), \prod_{i=1}^{n}(\beta_i), \prod_{i=1}^{n}(\gamma_i), \prod_{i=1}^{n}(\delta_i)\right]$$

Furthermore, consider the parallel system shown in fig. (1.1.4), where the reliability of the subsystem P_i is represented by the fuzzy number $\tilde{R}_i$ shown in fig. (1.1.1).

The reliability of parallel system can be evaluated and is equal to:

$$
\begin{aligned}
1-\prod_{i=1}^{n}\left(1-\tilde{R}_i\right) &= 1-\prod_{i=1}^{n}\left[1-\left(\alpha_i,\, \beta_i,\, \gamma_i,\, \delta_i\right)\right] \\
&= 1-\Big\{\left[\left(1-\alpha_1\right), \left(1-\beta_1\right), \left(1-\gamma_1\right)\left(1-\delta_1\right)\right]\otimes \\
&\qquad \left[\left(1-\alpha_2\right), \left(1-\beta_2\right), \left(1-\gamma_2\right)\left(1-\delta_2\right)\right]\otimes \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\qquad \left[\left(1-\alpha_1\right), \left(1-\beta_2\right), \left(1-\gamma_3\right)\left(1-\delta_4\right)\right]\Big\} \\
&= (1,1,1,1)-\left(\prod_{i=1}^{n}\left[1-\left(\alpha_i\right)\right], \prod_{i=1}^{n}\left[1-\left(\beta_i\right)\right], \prod_{i=1}^{n}\left[1-\left(\gamma_i\right)\right], \prod_{i=1}^{n}\left[1-\left(\delta_i\right)\right]\right) \\
&= \left(1-\prod_{i=1}^{n}\left[1-\left(\alpha_i\right)\right],\ 1-\prod_{i=1}^{n}\left[1-\left(\beta_i\right)\right],\ 1-\prod_{i=1}^{n}\left[1-\left(\gamma_i\right)\right],\ 1-\prod_{i=1}^{n}\left[1-\left(\delta_i\right)\right]\right) \qquad (5.7)
\end{aligned}
$$

A technical example:

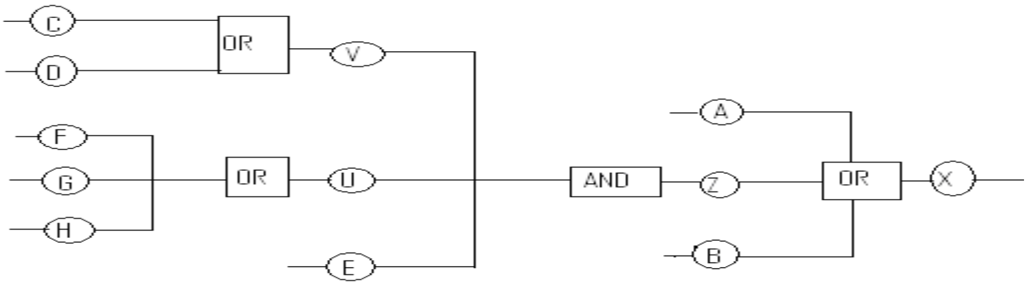

Figure 1.1.6: Fault tree of the example

Two grinding machines are working next to each other. What is the possibility that people coming in the vicinity of the machines are injured mainly by getting a chip into the eye? It is obvious that the most endangered persons are the operators, who are obliged to wear safety glasses but often fail to do this. Furthermore, endangered are persons coming in the vicinity of the machines, bringing and carrying away items, and those entering the area for other reasons.

Table 1.1.1: The basic events contributing to the accident

Symbol	Basic events
A	Operator 1 fails to wear safety glasses.
B	Operator 2 fails to wear safety glasses.
C	Machine 1 is operating.
D	Machine 2 is operating.
E	Persons entering the area without safety glasses.
F	Persons entering the endangered area bringing material.
G	Persons entering area carrying away made product.
H	Persons entering the area for other reasons.

The fault tree for the main event that somebody will be injured can be constructed as shown in fig. 1.1.6.

The basic events contributing to the accident are summarized in table [1.1.1].

Assume that the basic events are mutually independent and reliability of the basic events is represented by trapezoidal fuzzy numbers parameterized by $(\alpha_i, \beta_i, \gamma_i, \delta_i)$. Then we can see that

$$\tilde{R}_A = (0.00888, 0.01444, 0.02, 0.0255)$$
$$\tilde{R}_B = (0.00888, 0.01444, 0.02, 0.0255)$$
$$\tilde{R}_C = (0.75552, 0.77776, 0.8, 0.82224)$$
$$\tilde{R}_D = (0.75552, 0.77776, 0.8, 0.82224)$$
$$\tilde{R}_E = (0.94434, 0.97217, 1.0, 1.02783)$$
$$\tilde{R}_F = (0.04722, 0.04861, 0.05, 0.05139)$$
$$\tilde{R}_G = (0.04722, 0.04861, 0.05, 0.05139)$$
$$\tilde{R}_H = (0.0944, 0.00972, .000974, 0.009776)$$

From fig. 6, the truth function of the main event X can be written as follows:

U=F+G+H

V=C+D

Z=E+U+V

X=A+B+Z

From equation (5.6) and (5.7), we can get

$$\begin{aligned}\tilde{R}_U &= 1 - (1 - \tilde{R}_F) \otimes (1 - \tilde{R}_G) \otimes (1 - \tilde{R}_H) \\ &= 1 - [0.8220635, 0.89344942, 0.89370243, 0.89063891] \\ &= [0.10936109, 0.10629757, 0.10365505, 0.1779365]\end{aligned}$$

$$\begin{aligned}\tilde{R}_V &= 1 - (1 - \tilde{R}_C) \otimes (1 - \tilde{R}_D) \\ &= 1 - [0.05977, 0.0493906, 0.04, 0.031598617] \\ &= [0.96840138, 0.96, 0.9506094, 0.94023]\end{aligned}$$

$$\begin{aligned}\tilde{R}_Z &= \tilde{R}_E \otimes \tilde{R}_U \otimes \tilde{R}_V \\ &= [0.100010, 0.099205736, 0.0985354, 0.0169438]\end{aligned}$$

$$\begin{aligned}\tilde{R}_X &= 1 - (1 - \tilde{R}_A) \otimes (1 - \tilde{R}_B) \otimes (1 - \tilde{R}_c) \\ &= 1 - [0.8840771, 0.8749671, 0.8657665, 0.5447422] \\ &= [0.115922, 0.125038, 0.1342334, 0.45525]\end{aligned}$$

It is obvious that the above results coincide with the one presented in. However, from the above procedure, we can see that the proposed method has the advantage of low computation complexity compared to.

Concluding Remarks:

Reliability of each system component involves uncertainty in the performance of the system and its output. This kind of uncertainty is referred as fuzziness. While the probability theory characterizes randomness, fuzzy set theory deals with fuzziness. PROFUST reliability theory provides more realistic estimates than PROBIST reliability theory. In this paperr, we have developed a new method to analyze fuzzy system reliability using fuzzy number arithmetic operations and fuzzy logic operators. The reliability of each system component is represented by a trapezoidal fuzzy number. The proposed method uses simplified fuzzy arithmetic operations of fuzzy numbers rather than complicated interval arithmetic operations of fuzzy numbers. Execution of the developed method is faster than the one presented in the literature.

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REVIEW: L^1 - CONVERGENCE OF MODIFIED TRIGONOMETRIC SUMS

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Abstract:

In this paper, review of L^1- convergence of real and complex trigonometric series with modified trigonometric sums is done as the modified sums converge to their limits better than classical sums.

Keywords: Conjugate Dirichlet kernel, Dirichlet kernel, Fezer Kernel, L^1-convergence, modified sums.

Introduction:

Let $h(x) = \frac{a_0}{2} + \sum_{p=1}^{\infty} a_p \cos px$ ... (1) and $g(x) = \sum_{p=1}^{\infty} b_p \sin px$ ... (2) be the trigonometric cosine and sine series. Then $S_n(x) = \sum_{p=1}^{n} a_p \cos px$ and and $\tilde{S}_n(x) = \sum_{p=1}^{n} b_p \sin px$ be the partial sums of the series (1) and (2) respectively, where a_0, a_1,a_2,…and b0, b_1, b_2,…are real or complex coefficients.

Convergence in L_1 norm:

The series (1) is said to converge in $L_1(0, \pi)$ norm if $\|f - S_n\| = 0(1), n \to \infty$, where we denote $\|f\| = \int_0^{\pi} |f|\ dz$ where L is metric space.

Convex Sequence:

([1], Vol. I, p. 4) A sequence $\{\beta_k\}$ is said to be convex if $\Delta^2\beta_k \geq 0$ Where $\Delta\beta_k = \beta_k - \beta_{k+1}$ and $\Delta^2\ \beta_k = \Delta\beta_k - \Delta\beta_{k+1}$

Quasi- Convex Sequence:

([1],Vol.II,p.202) A sequence $\{\beta_k\}$ is said to be Quasi- convex if

$$\sum_{k=1}^{\infty}(k+1)\left|\Delta^2\beta_k\right|<\infty$$

A sequence $\{\alpha_k\}$ is said to be Generalized Quasi- Convex if $\sum_{k=1}^{\infty} k^r\left|\Delta^2\beta_k\right|<\infty,\ \ r=0,1,2,3\ldots.$

Class S (Sidon-Telyakovskii class):

([4], [5]) A sequence $\{\beta_k\}$ is said to follow class S if β_k = 0(1), as $k\to\infty$ and there, exists a sequence, $\{A_k\}$ such that

(i) $A_k\ \downarrow 0$ as $k\to\infty$ (ii) $\sum_{k=0}^{\infty} A_k\ <\infty$ (iii) $\left|\beta_k\right|\ \leq A_k\ \ \forall$ k.

After that class Sr, r = 1,2,... is invented by Tomovski [6], which is defined as: A sequence is said to fit class Sr, if ak→0 , as k →∞ and if there exists a, monotonically decreasing sequence $\{A_k\}$ which satisfies $\sum_{k=1}^{\infty} K^r A_k\ <\infty$ and $|\Delta ak|\leq Ak$, for all k. As A_k $\downarrow$ 0 and $\sum_{k=1}^{\infty} K^r A_k\ <\infty$ It implies $K^{r+1}A_k=0(1)\,, K\ \to\infty$ .Clearly S_{r+1} is subset of S_r $\forall$ r = 1,2.3 ... and S_0 = S

Class S*:

A sequence $\{\beta_k\}$ is said to follow class S* if $\beta_k=0(1), k\to\infty$ and there exists a sequence $\{A_k\}$ such that

(i) $\{A_k\}$ is quasi-monotone (ii) $\sum_{k=0}^{\infty} A_k\ <\infty$ (iii) $\left|\beta_k\right|\ \leq A_k\ \ \forall$ k

Class R:

A null sequence $\{\beta_k\}$ is said to follow the class R if $\sum_{k=1}^{\infty} k^2\left|\Delta^2\left(\frac{\beta_k}{k}\right)\right|\ <\infty$

Class K:

If $\beta_k=0(1)\,,k\ \to\infty$ and $\sum_{k=1}^{\infty} k\left|\Delta^2\beta_{k-1}-\Delta^2\beta_{k+1}\right|\ <\infty$ $(\beta_{0=}\ 0)$ then we will say that $\{\beta_k\}$ belongs to class K.

Class K$^\alpha$:

If $\beta_k=0(1)\,,k\ \to\infty$ and $\sum_{k=1}^{\infty} k^{\alpha}\ |\Delta^{\alpha+1}\beta_{k-1}-\Delta^{\alpha+1}\beta_{k+1}|\ <\infty$ $(\beta_{0=}\ 0)\ for\ \alpha>0$

Then we will say that $\{\beta_k\}$ is related to class K^α.

Class J:

A null sequence $\{\beta_k\}$ of positive numbers is follow the class J if there exists a sequence , $\{A_k\}$ such that

(i) $A_k \downarrow 0$ as $k \to \infty$ (ii) $\sum_{k=0}^{\infty} kA_k < \infty$ (iii) $\left|\Delta\left(\frac{\beta_k}{k}\right)\right| \leq \frac{A_k}{k}\ \forall\, k.$

Young [2] and ,Kolmogorov [3] initiated the research on L_1-convergence of trigonometric series underneath some special coefficients with classes of convex and ,quasi-convex Sequences .

Theorem 1.([2], [3]):

If $\{a_k\}\downarrow 0$ and $\{a_k\}$ is convex or even quasi-convex, then if and only if, condition for L_1 convergence ,of series (1) is that, $a_k \log k = 0(1),\ ,k \to \infty$. Theorem 1 for the cosine series (1) with coefficients satisfying the class S is indiscriminated by Telyakovskii [5] in the following way:

Theorem 2. [5]:

Let $\{a_k\}$ be the, sequence of the series (1) ,belonging to class S, then if and only if, condition for L_1-convergence of, (1) is that $a_k \log k. = 0(1),\ ,k \to \infty$ L_1-convergence of trigonometric series with unique coefficients has been studied by numerous authors. During the literature survey, it could be observed that many authors have introduced improved trigonometric sums as their results for L_1 convergence of trigonometric series are better than results of classical sums.

Rees and Stanojevic [7] have introduced modified cosine sum as

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_j)\ \cos kx$$

L_1-convergence of above defined cosine sum under different set of conditions on the coefficients are studied by Garrett and Stanojevic [18], Singh and Sharma[20]. Ram [19], proved the L_1-convergence of above defined cosine sum when the coefficients of Fourier series belong to class S.

Kumari and Ram ([8], [9]) introduced new modified cosine sum as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

They discussed the L^1-convergence of above defined cosine sum when coefficients are under class S and class R. Kaur and Bhatia proved the L^1-convergence of above defined cosine sum results for class S*. Singh and Modi proved the same results by eliminating the condition $a_n \log n = 0(1)$ as $n \to \infty$.

Kaur[9] introduced a new modified sine sums as

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}\left(\Delta a_{j-1} - \Delta a_{j+1}\right)\sin kx$$

and studied the L^1-convergence of this modified sine sum with semi-convex coefficients.Also, **Kaur et al. [12]** introduced a new class of numerical sequences as class K and proved that the sequence $\{a_k\}$ belong to the class K, then $K_n(x)$ converges to f(x) in the L^1-norm.

Singh and Kaur [19] defined new modified generalised sine sums

$$K_{nr}(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}(\Delta^r a_{k-1} - \Delta^r a_{k+1})\tilde{S}_k^{\,r-1}(x)$$

and a new class of coefficient K^α and proved that if the sequence $\{a_k\}$ belong to the class K^α, then $K_{nr}(x)$ converges to f(x) in the L^1-norm

Hooda, Ram and Bhatia [10] informed about,a new modified cosine sums, as

$$f_n(x) = \frac{1}{2}\left(a_1 + \sum_{k=0}^{n}\Delta^2 a_k\right) + \sum_{k=1}^{n}\left(a_{k+1} + \sum_{j=k}^{n}\Delta^2\ a_j\right)\cos kx$$

and studied its L^1 convergence and a result of Telyakovskii [21] is deduced as a corollary.

Kaur, Bhatia and Ram [12] introduced new modified cosine sums as

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}\left(\Delta b_{j-1} - \Delta b_{j+1}\right)\cos kx$$

and have studied the L^1-convergence of modified sine sums under a different class.J

N.L.Braha and Xrasniqi introduced new modified sums

$$N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n}\sum_{j=k}^{n}\left(\Delta^2 a_{j-1} - \Delta^2 a_j\right)\cos kx + \frac{a_1}{\left(2\sin\left(\frac{x}{2}\right)\right)^2}$$

and discussed L^1-convergence under semi-convex coefficients.

Krasniqi [15] have introduced new modified cosine and sine sums as

$$H_n(x) = -\frac{1}{2\sin x}\sum_{k=0}^{n}\sum_{j=k}^{n}\Delta\left[\left(a_{j-1} - a_{j+1}\right)\cos jx\right]$$

and studied the L^1-convergence of these modified cosine and sine sums with semi convex coefficients.

Krasniqi [16] have invented new modified cosine sum as

$$G_n(x) = \frac{a_0}{2} + \sum_{k_1=1}^{n} \sum_{k_2=k_1}^{n} \sum_{k_3=k_2}^{n} \Delta^2 (a_{k_3} \cos k_3 x)$$

where $\Delta^2 a_k = \Delta(\Delta a_k) = \Delta(a_k - a_{k+1}) = a_k - a_{k+1} - (a_{k+1} - a_{k+2}) = a_k - 2a_{k+1} + a_{k+2}$

Chouhan ,Kaur and Bhatia [17] have introduced new modified cosine and sine sum

$$f_n(x) = \sum_{k=1}^{n}\left[\sum_{j=k}^{n}\left(\Delta a_{j+1} + \sum_{i=j}^{n} \Delta^3 a_i\right)\right] \cos kx$$

$$g_n(x) = \sum_{k=1}^{n}\left[\sum_{j=k}^{n}\left(\Delta b_{j+1} + \sum_{i=j}^{n} \Delta^3 b_i\right)\right] \sin kx$$

and study the L^1-convergence of these modified cosine and sine sums under the class S. with the condition $n^2 a_n = 0(1)$ as $n \to \infty$.

Singh and Modi [18] have introduced new modified cosine and sine sums as

$$u_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j \cos jx}{2^j}\right) 2^k$$

$$v_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{b_j \sin jx}{2^j}\right) 2^k$$

and study the L^1- convergence of these modified cosine and sine sums under the class S.

Chouhan ,Kaur and Bhatia [19] have introduced new modified cosine and sine sum

$$f_n(t) = \sum_{k=1}^{n}\left(\frac{a_{k+1}}{k+1} + \sum_{j=k}^{n} \Delta^2\left(\frac{a_j}{j}\right)\right) k \cos kt \qquad (a_0 = a_1 = a_2 = 0)$$

and $$g_n(t) = \sum_{k=1}^{n}\left(\frac{b_{k+1}}{k+1} + \sum_{j=k}^{n} \Delta^2\left(\frac{b_j}{j}\right)\right) k \sin kt \qquad (b_1 = b_2 = 0)$$

and study the L^1- convergence of these modified sums when $\{a_k\}$ is generalized semi-convex.

Priyanka and Singh new modified cosine and sine sums as

$$R_n(x) = \sum_{k=1}^{n}\left[\left(\sum_{j=k}^{n}\left(\Delta\left(\frac{a_{j+1}}{j+1}\right) + \sum_{i=j}^{n} \Delta^3\left(\frac{a_i}{i}\right)\right)\right) k \cos kx\right]$$

Where $\Delta^3 a_k = \Delta^2 a_k - \Delta^2 a_{k+1} = a_k - 3a_{k+1} + 3a_{k+2} - a_{k+3}$

$$Q_n(x) = \sum_{k=1}^{n}\left[\left(\sum_{j=k}^{n}\left(\Delta\left(\frac{b_{j+1}}{j+1}\right) + \sum_{i=j}^{n} \Delta^3\left(\frac{b_i}{i}\right)\right)\right) k \sin kx\right]$$

Where $\Delta^3 b_k = \Delta^2 b_k - \Delta^2 b_{k+1} = b_k - 3b_{k+1} + 3b_{k+2} - b_{k+3}$

and study the L^1-convergence of these modified cosine and sine sums under the class S_2.

Conclusion:

In this review paper various types of modified trigonometric sums are studied under different set of conditions on the coefficient class. In all the above sums $\lim_{n\to\infty} a_n \log n = 0(1)$ is necessary and sufficient condition for L^1-convergence of real and complex trigonometric sine and cosine series. Still there is a need to formulate new modified sums and new set of coefficients under which this condition is eliminated.

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SOME PROPERTIES AND INVERSION THEOREM OF GENERALIZED MELLIN WHITTAKER TRANSFORM

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Abstract:

The aim of this paper is to study the properties of the generalized Mellin-Whittaker transform. In this paper we have also proved the inversion theorem and some lemmas related to inversion theorem.

Introduction:

For almost two centuries the method of function transformations has been used successfully in solving many problems in engineering, mathematical physics and applied mathematics. A function transformation simply means mathematical operation through which a real or complex valued function is transformed into another setting of data in which the original problem can be solved more easily or in which the problem has clear physical meaning. Integral transform is one of the techniques in the function transformation methods. The integral transform methods are of great importance in the initial and final values problems of partial differential equations. The theory of integral transform flourished and continues to do so since the birth of great mathematician Laplace. Number of integral transforms e.g. Fourier, Mellin, Hankel, Hilbert, Stieltjes etc. are developed as per the need arose in the physical situations or in partial differential equations. Mellin transform is also used in scale invariant systems [3, 4, 5, 7].

Extension of some transformations to generalized functions have been done from time to time and their properties have been studied

This chapter deals with different theorems concerned with generalized Mellin-Whittaker transform. we have derived different types of operation transform formulae for Mellin-Whittaker transform which will be useful to solve boundary value problems in partial differential equations. Properties like shifting, scaling, differentiation, translation of a Mellin variable are discussed.

Generalized Mellin-Whittaker Transform:

For $f(x,t) \in MW'$(dual space of $MW_{a,b}$) where $a = (a_1, a_2)$ and $b = (b_1, b_2)$, $a_2 < 1$ and $b_2 > m + k + ½$ when $0 < x < \infty$, $0 < t < \infty$ and $(s,y) \in \Omega_f = \{ (s,y) / a_1 < Res < b_1 \; and \; Re[(p+q)y] > 0 \}$ the function $F(s,y)$ is defined as the generalized Mellin-Whittaker transform,

$$F(s,y) = MWf(x,t) = \langle f(x,t), \varphi(x,t,s,y) \rangle,$$

where $\varphi(x,t,s,y) = x^{s-1} e^{-\frac{q}{2}yt} (yt)^{m-\frac{1}{2}} W_{k,m}(pyt).$

The right hand side is meaningful because according to theorem 3.5.2, $\varphi(x,t,s,y) \in MW$ and $f(x,t) \in MW'$.

Moreover if $f(x,t)$ is a regular generalized function then conventionally we can write,

$$F(s,y) = [MWf(x,t)](s,y)$$

$$= \int_0^\infty \int_0^\infty x^{s-1} e^{-\frac{q}{2}yt} (yt)^{m-\frac{1}{2}} W_{k,m}(pyt) f(x,t) dx dt.$$

Properties of Generalized Mellin-Whittaker Transform:

In this section we derive some properties of generalized Mellin-Whittaker transform. Next in all the properties we suppose $f(x,t) \in MW'$ and we listed the proved properties in tabular form in the next subsection.

Table 1: Properties of the Mellin-Whittaker Transform

Sr. No.	Function	Mellin-Whittaker Transform
1	$\frac{d}{dx} f(x,t)$	$-(s-1)(MW\, f(x,t))(s-1,y)$
2	$\frac{d^k}{dx^k} f(x,t)$	$(-1)^k (s-k)! \left(MW \frac{1}{x^k} f(x,t) \right)(s,y)$
3	$x.f_x(x,t)$	$-s\left[MW\, f(x,t) \right](s,y)$
4	$f_t(x,t)$	$\frac{1}{\sqrt{p}} \left[MW_{k+1/2, m-1/2} f(x,t) \right](s,y) - \left(\frac{p-q}{2} \right) y \left[MW_{k,m} f(x,t) \right](s,y)$

5	$f_{x,t}(x,t)$	$(s-1)\frac{1}{\sqrt{p}}\left[MW_{k+1/2,m-1/2}\frac{1}{x}f(x,t)\right](s,y)-(s-1)$ $\left(\frac{p-q}{2}\right)y\left[MW_{k,m}\frac{1}{x}f(x,t)\right](s,y)$
6	$f_{xx,tt}(x,t)$	$(s-1)(s-2)\left\{\frac{1}{p}\left[MW_{k+1,m-1}\frac{1}{x^2}f(x,t)\right](s,y)-2\frac{1}{\sqrt{p}}\left(\frac{p-q}{2}\right)y\right.$ $\left[MW_{k+\frac{1}{2},m-\frac{1}{2}}\frac{1}{x^2}f(x,t)\right](s,y)+\left(\frac{p-q}{2}\right)^2y^2$ $\left.\left[MW_{k,m}\frac{1}{x^2}f(x,t)\right](s,y)\right\}$
7	$f(\alpha x,t)$	$\frac{1}{\alpha^s}\left[\ MW\ f(x,t)\ \right],\qquad \alpha\ \varepsilon\ R^+$
8	$f(x,\beta t)$	$\frac{1}{\beta}\left[\ MW\ f(x,t)\ \right]\left(s,\frac{y}{\beta}\right),\qquad \beta\ \varepsilon\ R^+$
9	$f(\alpha x,\beta t)$	$\frac{1}{\beta.\alpha^s}\left[\ MW\ f(x,t)\ \right]\left(s,\frac{y}{\beta}\right),\qquad \alpha,\beta\ \varepsilon\ R^+$
10	$x^\alpha f(x,t)$	$F\left[\ (s+\alpha),y\right],\qquad \alpha\ \varepsilon\ R$
11	$f(x^n,t)$	$n^{-1}F\left(\frac{s}{n},y\right),\qquad n\ \varepsilon\ N$
12	$f(x^{-n},t)$	$n^{-1}F\left(\frac{-s}{n},y\right),\qquad n\ \varepsilon\ N$
13	$f\left(\frac{1}{x},t\right)$	$F(-s,y)$
14	$\log x\, f(x,t)$	$\frac{d}{ds}F(s,y)$
15	$x\, f_x(x,t)$	$-s\left[\ MW\ f(x,t)\ \right](s,y)$
16	$x^k f_x^{\ k}(x,t)$	$(-1)^k\frac{(s+k-1)!}{(s-1)\,!}\left[\ MW\ f(x,t)\ \right](s,y)$

Inversion Theorem:

This section is devoted for the derivation of an inversion theorem for Mellin-Whittaker Transform. Inversion theorem requires two lemmas which have been proved in this section.

Lemma: If $f(u,z) \in MW', \Phi(yt) = \dfrac{1}{2\pi i}\int_{-\infty}^{\infty} \dfrac{(yt)^{-v}}{\Psi(1-v)}\, dv,\ o < r < \infty$

where

$$\psi(v) = \frac{p^{m+\frac{1}{2}}\Gamma(m \pm m + v)}{\left[\frac{1}{2}(q+p)\right]^{m+m+v}\Gamma(m-k+v+\frac{1}{2})} \times {}_2F_1\left[\begin{matrix} m+m+v, m-k+\dfrac{1}{2}; \\ \dfrac{q-p}{q+p} \\ m-k+v+\frac{1}{2}; \end{matrix}\right]$$

then

$$\int_{-r}^{r}\int_{o}^{\infty} \left\langle f(u,z),\ u^{S-1} e^{-\frac{q}{2}yz} (yz)^{m-\frac{1}{2}} W_{k,m}(pyz)\right\rangle \left\{\int_{o}^{\infty}\int_{o}^{\infty} \Phi(yt) x^{-s} \phi\ (x,t,s,y) dxdt\right\} dyds =$$

$$\left\langle f(u,z), \int_{-r}^{r}\int_{o}^{\infty}\int_{o}^{\infty}\int_{o}^{\infty} \Phi(yt)x^{-s}\ u^{s-1} e^{-\frac{q}{2}uz} (yz)^{m-\frac{1}{2}} W_{k,m}(pyz)\phi\ (x,t,s,y) dxdtdyds\right\rangle.$$

Proof: If $\phi\ (x,t,s,y) = 0$ there is nothing to prove.

So, assume $\phi\ (x,t,s,y) \neq 0$

To prove the lemma means to show –

$$\int_{-r}^{r}\int_{o}^{\infty} F(s,y) \left\{\int_{o}^{\infty}\int_{o}^{\infty}\ \Phi(yt)x^{-s}\phi\ (x,t,s,y)\ dxdt\right\}\ dsdy$$

$= \langle f(u,z), \int_{-r}^{r}\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} F(u,z,s,y)\Phi(yt)x^{-s}\phi(x,t,s,y)dxdtdsdy\rangle.$

Now, note that $F(s,y)$ is analytic for $a_2 < \sigma + 1 - m,\ |\ Re\ s\ | < c.$

Moreover, $\int_{o}^{\infty}\int_{o}^{\infty} \Phi(yt)x^{-s}\phi\ (x,t,s,y) dxdt$ is entire function.

Therefore integral on l.h.s. definitely exists.

Now to show that r.h.s is meaningful, we first show that,

$$\int_{-r}^{r}\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} F(u,z,s,y)\Phi(yt)x^{-s}\phi(x,t,s,y)dxdtdsdy = A(u,z) \in MW$$

and $f(u,z) \in MW'$

Hence r.h.s. will be meaningful.

Now consider,

$$\gamma_{k,l}[A(u,z)] = sup\left| \lambda(u,z) u^{k+1} D_u^k (zD_z)^l \{ A(u,z)\} \right| \quad (4.1.1)$$

Since $\sup \left| \lambda\ (u,z) u^{k+1} D_u^k (-zD_z)^l \phi\ (u,z,s,y) \right| < \infty$

and $\int_o^\infty \int_o^\infty \Phi(yt) x^{-s} \phi\ (x,t,s,y)\ dxdt$ is entire.

Therefore r.h.s. of equation (4.1.1) $< \infty \ \Rightarrow A(u,z) \in MW$.

Next partition the path of integration from $s = c - ir$ to $s = c + ir$ into m-intervals each of length $\frac{2r}{m}$ and partition the path of integration from $y = a - ib$ to $y = a + ib$, into n-intervals each of length $\frac{2b}{n}$.

Again let $s_k = c + iw_k$ and $y_j = a + iB_j$ be any point in the k^{th} and j^{th} interval respectively.

Define $\theta_{m,n}(u,z) = \sum_{j=1}^{n} \sum_{k=1}^{m} u^{s_k - 1} \frac{y_j^{-v}}{\Psi(1-v)} \left\{ \int_o^\infty \int_o^\infty x^{-s_k} t^{-V} \phi\ (x,t,s,y) dxdt \right\} e^{\frac{-q}{2} y_j z}$

$$(y_j z)^{m-\frac{1}{2}} W_{k,m}(p y_j z) \frac{2r}{m} . \frac{2b}{n}.$$

Now applying $f(u,z)$ to $\theta_{m,n}(u,z)$ term by term we get,

$$\left\langle f(u,z), \theta_{m,n}(u,z) \right\rangle = \left\langle f(u,z),\ \sum_{j=1}^{n} \sum_{k=1}^{m} u^{s_k - 1} \frac{y_j^{-v}}{\Psi(1-v)} \left\{ \int_o^\infty \int_o^\infty x^{-s_k} t^{-V} \phi\ (x,t,s,y) dxdt \right\} e^{\frac{-q}{2} y_j z} \right.$$

$$\left. (y_j z)^{m-\frac{1}{2}} W_{k,m}(p y_j z) \frac{2r}{m} . \frac{2b}{n} \right\rangle$$

$$= \int_{-b}^{b} \int_{-r}^{r} \left[\left\langle f(u,z),\ u^{s-1} e^{\frac{-q}{2} yz} (yz)^{m-\frac{1}{2}} W_{k,m}(pyz) \right\rangle \right.$$

$$\left. \int_o^\infty \int_o^\infty \frac{y^{-v} t^{-v}}{\Psi(1-v)} x^{-s} \phi\ (x,t,s,y)\ dxdt \right] dyds$$

as $m \to \infty$ and $n \to \infty$.

In view of the fact that

$$\left\langle f\ (u,z),\ u^{s-1}\ \ e^{\frac{-q}{2} yz} (yz)^{m-\frac{1}{2}} W_{k,m}(pyz) \right\rangle \int_o^\infty \int_o^\infty \frac{y^{-v} t^{-v}}{\Psi(1-v)} x^{-s} \phi\ (x,t,s,y)\ \ dxdt$$

is continuous function of s and y.

Since $f(u,z) \in MW'$, all that remains to be proven is that $\theta_{m,n}(u,z)$ converges in MW to,

$$\int_{-b}^{b}\int_{-r}^{r} u^{s-1} e^{-\frac{q}{2}yz}(yz)^{m-\frac{1}{2}} W_{k,m}(pyz)\frac{y^{-v}t^{-v}}{\psi(1-v)} x^{-s}\varphi(x,t,s,y)dsdy = B(u,z) \text{ say}$$

For which we show that $|\,\theta_{m,n}(u,z) - B(u,z)\,| \to 0$ uniformly on $o < u < \infty$, $o < z < \infty$ as $m \to \infty$ and $n \to \infty$.

Consider

$$\xi_{k,l}\left[\theta_{m,n}(u,z) - B(u,z)\right] = \sup\left|\lambda(u,z)u^{k+1}D_u^k(-zD_z)^l\left\{z\left[\theta_{m,n}(u,z) - B(u,z)\right]\right\}\right|$$

$$\sup\left|\lambda(u,z)u^{k+1}D_u^k(-zD_z)^l\left[\left\{z\sum_{j=1}^{n}\sum_{k=1}^{m} u^{s_k-1}\frac{y_j^{-v}}{\psi(1-v)}\left(\int_o^\infty\int_o^\infty x^{-s_k}t^{-v}\phi(x,t,s,y)dxdt\right)\right.\right.\right.$$

$$\left. e^{\frac{-q}{2}y_jz}(y_jz)^{m-\frac{1}{2}}W_{k,m}(py_jz)\frac{2r}{m}\frac{2b}{n}\right\} - \left\{z\int_{-b}^{b}\int_{-r}^{r} u^{s-1}\; e^{\frac{-q}{2}yz}(yz)^{m-\frac{1}{2}}W_{k,m}(pyz)\right.$$

$$\left.\left.\left.\frac{y^{-v}t^{-v}}{\Psi(1-v)}x^{-s}\phi(x,t,s,y)dyds\right\}\right]\right| \qquad (4.1.2)$$

But $\sup\left|\lambda(u,z)u^{k+1}D_u^k(-zD_z)^l\left\{z\,u^{s_k-1}\,e^{\frac{-q}{2}y_jz}(y_jz)^{m-\frac{1}{2}}W_{k,m}(pyz)\right\}\right| < M_k$ (say) and

$$\sup\left|\lambda(u,z)u^{k+1}D_u^k(-zD_z)^l\left\{z\,u^{s-1}e^{\frac{-q}{2}yz}(yz)^{m-\frac{1}{2}}W_{k,m}(pyz)\right\}\right| < M$$

$\therefore$ There exist U and Z such that for $|\,u\,| > U$ and $|\,z\,| > Z$ such that,

$$M < \frac{\varepsilon}{3}\left[\int_o^\infty\int_o^\infty \frac{y^{-v}t^{-v}x^{-s}}{\psi(1-v)}\;\phi\left(x,t,s,y\right)\;dxdt\right]^{-1}$$

$$M_k < \frac{\varepsilon}{3}\left[\int_o^\infty\int_o^\infty \frac{y_j^{-v}t^{-v}}{\psi(1-v)}x^{-s_k}\phi\left(x,t,s,y\right)\;dxdt\right]^{-1}.$$

First term on r.h.s. of equation (3.7.2) is bounded by

$$\frac{\varepsilon}{3}\left[\int_o^\infty\int_o^\infty \frac{y^{-v}t^{-v}x^{-s}}{\psi(1-v)}\;\phi\,(x,t)dxdt\right]^{-1}\sum_{j=1}^{n}\sum_{k=1}^{m}\left|\frac{y_j^{-v}t^{-v}}{\psi(1-v)}x^{-s_k}\right|\frac{2r}{m}\frac{2b}{n} \qquad (4.1.3)$$

Choose m and n, so large say $m > m_o$ and $n > n_o$ that expression (4.1.3) will be less than $\frac{2\varepsilon}{3}$.

Thus, $\gamma_{k,l}[\,\theta_{m,n}(u,z) - B(u,z)\,] < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

But ε is arbitrary. This implies that,

$\theta_{m,n}(u,z) \to B(u,z)$ uniformly for $|u| > U$ and

$|z| > Z$ for $m > m_o$ and $n > n_o$.

Also since $\lambda(u,z)u^{k+1}D_u^k(-zD_z)^l \left\{ z\, u^{s-1} e^{\frac{-q}{2}yz} (yz)^{m-\frac{1}{2}} W_{k,m}(pyz) \right\}$ is

uniformly continuous function of u and z in $|u| \le U$ and $|z| \le Z$.

Therefore there exist m_1 and n_1 integers such that $m > m_1$ and $n > n_1$,

$$\gamma_{k,l}[\,\theta_{m,n}(u,z) - B(u,z)\,] \to 0 \text{ for } |u| \le U \text{ and } |z| \le Z.$$

Thus when $M = \max(m_0, m_1)$ and $N = \max(n_0, n_1)$ then for $m \ge M$, $n \ge N$.

$$\gamma_{k,l}[\theta_{m,n}(u,z) - B(u,z)] \to 0 \text{ uniformly in MW,}$$

for $-\infty < u < \infty$ and $-\infty < z < \infty$.

Lemma: If

$$\frac{1}{\pi^2}\int_o^\infty\int_o^\infty \left[\left(\frac{z}{t}\right)^{d-1} \frac{\sin r \log z/t}{t \log z/t} \left(\frac{u}{x}\right)^{\sigma-1} \frac{\sin w \log u/x}{x \log u/x}\right] \phi\,(x,t)\,dx\,dt = B\,(u,z)$$

Then $B\,(u,z) \to \phi\,(u,z)$ in MW as $r \to \infty$ and $w \to \infty$.

Proof: For this we have to show that,

$$\xi_{k,l}\left[B\,(u,z) - \phi\,(u,z)\right] = \sup\left|\lambda(u,z)u^{k+1}D_u^k\left(-zD_z\right)^l \left\{ z\left[\frac{1}{\pi^2}\int_o^\infty\int_o^\infty \left(\frac{z}{t}\right)^{d-1} \frac{\sin r \log z/t}{t \log z/t} \left(\frac{u}{x}\right)^{\sigma-1} \frac{\sin w \log u/x}{x \log u/x}\right] \phi\,(x,t)\,dx\,dt - \phi\,(u,z)\right\}\right| \to 0$$

Now $\displaystyle\int_o^\infty \frac{\sin r \log z/t}{t \log z/t}\,dt = \int_{-\infty}^\infty \frac{\sin r t_1}{t_1}\,dt_1 = \pi$, as $r \to \infty$

$$\int_o^\infty \frac{\sin w \log u/x}{x \log u/x}\,dx = \int_{-\infty}^\infty \frac{\sin w t_2}{t_2}\,dt_2 = \frac{\pi}{2} \text{ as } w \to \infty$$

$$\xi_{k,l}\left[B\,(u,z) - \phi\,(u,z)\right] = \sup\left|\lambda(u,z)u^{k+1}D_u^k\left(-zD_z\right)^l \left\{ z\left[\frac{1}{\pi^2}\int_o^\infty\int_o^\infty \left(\frac{z}{t}\right)^{d-1} \frac{\sin r \log z/t}{t \log z/t} \left(\frac{u}{x}\right)^{\sigma-1} \frac{\sin w \log u/x}{x \log u/x}\right] \phi\,(x,t)\,dx\,dt\right.\right.$$

$$-\frac{1}{\pi^2}\int_o^\infty\int_o^\infty\left[\frac{\sin r\log z/t}{t\log z/t}\frac{\sin w\log u/x}{x\log u/x}\right]\phi\,(u,z)\Bigg|$$

$$=\sup\Bigg|\lambda_{u,z}(u,z)u^{k+1}D_u^k(zD_z)^l\left\{z\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{z}{t}\right)^{d-1}\frac{\sin rt_1}{t_1}\left(\frac{u}{x}\right)^{\sigma-1}\frac{\sin wt_1}{t_1}\phi\left(ue^{-t_2},ze^{-t_1}\right)\right.$$

$$dt_1dt_2-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(u,z)\frac{\sin rt_1}{t_1}\frac{\sin wt_2}{t_2}dt_1dt_2\Bigg|$$

Since ϕ is continuous, as in Zemanian [85] p.66 it can be shown that,

$$\xi_{k,l}\left[B(u,z)-\phi\,(u,z)\right]\to 0$$

Hence the theorem is proved.

Inversion Theorem:

Let $f\in MW',\phi\in D(I)$ and Mellin-Whittaker transform of f is

$$F(s,y)=\left\langle f(x,t),\ x^{s-1}\ e^{-\frac{q}{2}yt}(yt)^{m-\frac{1}{2}}W_{k,m}(pyt)\right\rangle \qquad \text{then}$$

$$\left\langle\int_o^\infty\int_o^\infty x^{-s}\Phi(yt)F(s,y)dsdy,\ \phi\,(x,t,s,y)\right\rangle=\left\langle\ f,\ \ \phi\ \right\rangle,$$

where $\Phi(t)=\dfrac{1}{2\pi i}\displaystyle\int_{d-i\infty}^{d+i\infty}\frac{t^{-v}}{\psi(1-v)}dv$

where,

$$\psi(\,v\,)=\frac{p^{m+\frac{1}{2}}\,\Gamma(\,m\pm m+v)}{\left[\frac{1}{2}(q+p)\right]^{m+m+v}\Gamma(\,m-k+v+\frac{1}{2})}\times\ {}_2F_1\left[\begin{matrix}m+m+v, \qquad m-k+\dfrac{1}{2};\\ \dfrac{q-p}{q+p}\\ m-k+v+\frac{1}{2};\end{matrix}\right]$$

provided the integral exists.

Thus $f(x,t)=\displaystyle\int_o^\infty\int_o^\infty\Phi(yt)F(s,y)\ \ x^{-s}dsdy.$

Proof: - Consider,

$$\left\langle\frac{1}{2\pi}\int_o^\infty\int_o^\infty x^{-s}\Phi(yt)F(s,y)dsdy,\ \phi\,(x,t,s,y)\right\rangle$$

$$= \begin{matrix}\lim \\ r\to\infty\end{matrix}\left\langle \frac{1}{4\pi^2 i}\int_{d-ir}^{d+ir}\int_o^\infty\int_o^\infty x^{-s}\,\frac{y^{-v}t^{-v}}{\psi(1-v)}F(s,y)dvdsdy,\ \phi\,(x,t,s,y)\right\rangle$$

$$= \begin{matrix}\lim \\ r\to\infty\end{matrix}\left\langle \frac{1}{4\pi^2}\int_{-r}^{r}\int_o^\infty\int_o^\infty \frac{x^{-s}y^{-v}t^{-v}}{\psi(1-v)}\left\langle f(u,z),u^{s-1}\ \ e^{\frac{-q}{2}yz}(yz)^{m-\frac{1}{2}}W_{k,m}(pyz)\right\rangle drdsdy,\ \phi\,(x,t,s,y)\right\rangle$$

$$= \begin{matrix}\lim \\ r\to\infty\end{matrix}\frac{1}{4\pi^2}\left\langle f(u,z),\int_o^\infty\int_o^\infty\ \int_o^\infty\int_o^\infty\int_{-r}^{r}\frac{y^{-v}t^{-v}}{\psi(1-v)}x^{-s}u^{s-1}\ \ e^{\frac{-q}{2}yz}(yz)^{m-\frac{1}{2}}W_{k,m}(pyz)\ \phi\,(x,t,s,y)drdxdtdyds\right\rangle$$

by lemma (4.1) ,(4.2)

Using the result, from [70]

$$\int_o^\infty\int_o^\infty\frac{y^{-v}}{\psi(1-v)}\ \ e^{\frac{-q}{2}yz}(yz)^{m-\frac{1}{2}}W_{k,m}(pyz)f(u,z)dydz=\int_o^\infty z^{v-1}f(u,z)dz \qquad (4.3.1)$$

$$\therefore\left\langle\frac{1}{2\pi}\int_o^\infty\int_o^\infty x^{-s}\Phi(yt)F(s,y)dsdy,\ \phi\,(x,t,s,y)\right\rangle=\begin{matrix}\lim \\ r\to\infty\end{matrix}\frac{1}{4\pi^2}\int_o^\infty\int_o^\infty\int_o^\infty\int_o^{a}\int_{-r}^{r}t^{-v}x^{-s}\phi\,(x,t,s,y)u^{s-1}$$

$$\left[\int_o^\infty\int_o^\infty\frac{y^{-v}}{\psi(1-v)}e^{\frac{-q}{2}yz}(yz)^{m-\frac{1}{2}}W_{k,m}(pyz)f(u,z)dydz\right]drdxdtdwdu.$$

Now by using (4.3.1),

$$= \begin{matrix}\lim \\ r\to\infty\end{matrix}\frac{1}{4\pi^2}\int_o^\infty\int_o^\infty\int_o^\infty\int_o^\infty\int_{-r}^{r}t^{-v}x^{-s}\phi\,(x,t,s,y)u^{s-1}\int_o^\infty z^{v-1}f(u,z)dzdrdxdtdudw \qquad (4.3.2)$$

Now if $v = d + ir$ and $s = c + iw$

$$\begin{matrix}\lim \\ r\to\infty\end{matrix}\int_{-r}^{r}\ t^{-v}z^{v-1}dr=\begin{matrix}\lim \\ r\to\infty\end{matrix}\int_{-r}^{r}\left(\frac{z}{t}\right)^{v}z^{-1}dr=2\left(\frac{z}{t}\right)^{d}\frac{\sin(r\log z/t)}{z\log(z/t)}$$

Similarly $$\begin{matrix}\lim \\ w\to\infty\end{matrix}\int_{-w}^{w}x^{-s}u^{s-1}dw=\begin{matrix}\lim \\ w\to a\end{matrix}\int_{-w}^{w}\left(\frac{u}{x}\right)^{s}u^{-1}dw=2\left(\frac{u}{x}\right)^{c}\frac{\sin(w\log u/x)}{u\log(u/x)}$$

$$=2\left(\frac{u}{x}\right)\frac{\sin(w\log u/x)}{u\log(u/x)}.$$

$\therefore$ equation (4.3.2) becomes

$$=\frac{1}{4\pi^2}\int_o^\infty\int_o^\infty\int_o^\infty\int_o^\infty\left[4\ \left(\frac{z}{t}\right)^{d-1}\frac{\sin r\log z/t}{t\log z/t}\left(\frac{u}{x}\right)^{c-1}\frac{\sin w\log u/x}{x\log u/x}\right]\phi(x,t,s,y)f(u,z)du.dz.dx.dt$$

(4.3.3)

By lemma (4.2)

$$\int_o^\infty\int_o^\infty\left[\frac{1}{\pi^2}\left(\frac{z}{t}\right)^{d-1}\frac{\sin r\log z/t}{t\log z/t}\left(\frac{u}{x}\right)^{c-1}\frac{\sin w\log u/x}{x\log u/x}\right]\phi(x,t,s,y)dx.dt\to\phi(u,z)$$

$\therefore$ (4.3.3) => $\int_{o}^{\infty}\int_{o}^{\infty} \phi\,(u,z) f(u,z) du.dz = \langle f(u,z),\ \phi\,(u,z)\rangle$.

Hence the proof.

Uniqueness Theorem:

In the theory of integral transforms the uniqueness theorem is of special importance because of this theorem the question of recovery of a function is possible from its transform. In the next theorem we have shown that if two functions have the same Mellin-Whittaker transform then they coincide almost everywhere.

Theorem:

If $F(s,y) = [\,MWf(x,t)\,](s,y)$ for $(s,y) \in \Omega_f$ and

$G(s,y) = [\,MWg(x,t)\,](s,y)$ for $(s,y) \in \Omega_g$.

If $F(s,y) = G(s,y)$, then $f(x,t) = g(x,t)$ in the sense of equality in $D'(I)$.

Proof: By inversion theorem,

$$f(x,t) - g(x,t) = \int_0^\infty \int_0^\infty \Phi(yt) x^{-s} [\,F(s,y) - G(s,y)] ds dy$$

$$= 0\ ,$$

as $F(s,y) = G(s,y)$

$\Rightarrow\ f = g$ in $D'(I)$.

This proves uniqueness.

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SCHRODINGER'S EQUATIONS AND MATHEMATICAL PHYSICS

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Abstract:

Mathematical physics deals with the study of development of mathematical methods for the formulation of physical theories and application to problems in physics. Schrodinger using de Broglie's idea of matter waves developed a precise mathematical theory which has received the name of wave Mechanics. The essential feature of this theory is the incorporation of the expression for the De Broglie wavelength into the general classical wave equation. By this means a wave equation for a moving particle is derived, which is known as Schrodinger's fundamental wave equation. Schrodinger wave equation is a mathematical expression describing the energy and position of the electron in space and time, taking into accounts the matter, wave nature of the electron inside an atom.

This Review article discusses about the Schrodinger's equations. This article provides the simplified derivation of Schrodinger's wave equations starting from wave mechanics.

Keywords: Schrodinger's equation, Mathematical physics.

Introduction:

There are several different branches of mathematical physics such as vector analysis, Partial differential equations, Classical mechanics, Quantum mechanics, Relativity, Statistical mechanics etc.,. Usually mathematical physics has been very much linked with differential equations, integral equations. Its approach can be adapted to applications in classical as well as quantum mechanics. Partial differential equations are used in different fields of science. There are several theories in the history of science. There exist equations, in most of the theories, describing those theories in order to carry out some calculation and get the results. The Schrodinger equation is mathematically described as a partial differential equation that is used in quantum mechanics to describe the quantum behavior and state of the changes in physical system. Schrodinger expressed De Broglie hypothesis relating to the wave behavior of matter in a mathematical form that is useful for solving physical problems without additional arbitrary assumptions.

Similar endeavor have been done earlier by some researchers. Schrödinger's equation is a fundamental or building block of quantum mechanics and finds several applications. Some of the significant applications are Schrödinger equation is used to determine the allowed or permissible energy levels of quantum mechanical systems like atoms, electrons, protons, neutrons etc. The associated wave function gives the probability of finding the particle at a certain position. The allowed energy levels of a particle constrained to a rigid box can also be determined using SE. Square well potential is a typical example or problem in quantum mechanics that exemplify differences between classical and quantum mechanical situations. SE is the way to determine the potential energy, rather allowed energy levels, of vibrating atoms and to analyze their motion [1]. Numerical methods have been very important in basic research on physics. Nowadays partial differential equations (PDEs) related to main stream problems involve the use of numerical solutions to PDEs [2]. In quantum mechanics, the Schrodinger equation describes how the quantum state of a physical state evolves over time. It is not a simple algebraic equation but, in general, a linear partial differential equation. The solutions to the Schrodinger equation describe not only molecular, atomic and subatomic systems but also macroscopic systems. The Schrodinger equation is easy to use for the hydrogen atom, but when the atomic number is increased, the numerical methods are more effective and facilitate the resolution of the problem. The Schrodinger equation is easy to use for the hydrogen atom, but when the atomic number is increased, the numerical methods are more effective and facilitate the resolution of the problem. This equation has limitations because it only applies to particles with low velocities [3]. A wave equation should be able to model atom and molecule in realistic way. There can be possible correspondence between classical mechanics and quantum mechanics [4]. An approximate solution of the radial Schrodinger equation is obtained with a generalized group of potentials in the presence of both magnetic field and potential effect using supersymmetric quantum mechanics and shape invariance methodology. The inclusion of the potential effect greatly affects the accuracy of the results [5].

Schrodinger's equations:

According to De Broglie's hypothesis, every moving material particle has a wave associated with it, whose wavelength is given by the relation $\lambda = \frac{h}{mv}$. By this means, a wave equation for a moving particle is derived, which is known as Schrodinger fundamental wave equation. In 1926, in order to explain the dual nature of matter, Schrodinger considered every material particle as to be equivalent to a wave packet, which moves with a certain group velocity and each individual wave, comprising a wave packet, moves with a certain phase velocity. The concept of wave packet demanded the existence of a certain guiding wave. The equation of this guiding wave was derived by Schrodinger and is of immense use in the problems concerned with almost all the domains of physics as well as chemistry. It is an equation that describes the behavior of the wave function associated with the atomic particles.

The motion of a particle of atomic dimensions cannot be described by Newton's laws of motion which are applicable to bodies of macroscopic dimensions. The behavior of particles of microscopic size is governed by the associated de – Broglie waves or the wave function$\psi\,(\vec{r},t)$, the variable quantity that characterizes the De Broglie waves is called the wave function. Wave in general associated with quantities that vary periodically is called wave function. The wave function ψ associated with mechanical system contains in itself all the relevant information about the behavior of the system and hence defines it completely. In other words, the probability of experimentally finding the particle described by the wave function at the point x, y, z at the time t is proportional to $|\psi|^2$. Thus $|\psi|^2$ i.e. the square of the magnitude (amplitude) of the wavefunction $\psi\,(\vec{r},t)$ gives the probability density of finding the physical system (particle or photon) at a particular place at a given time.

To describe the motion of the particle in space and time we must know the value of $\psi\,(\vec{r},t)$ at all times and for this purpose it is necessary to find a differential equation which controls the space time behavior of the wave function$\psi\,(\vec{r},t)$. The solution of such a differential equation will gives the possible motion of the particle. There are two equations, which are time-dependent Schrodinger equation and a time-independent Schrodinger equation. Hence the necessary, differential equations are

Schrodinger Time Independent Wave Equation (Stationary state):

A state is said to be in a stationary state, if the function explaining the state does not include time. Thus the function $\psi\,(x,y,z)$ is a stationary state whereas function $\psi\,(x,y,z,t)$ is not. If a moving particle has a wave associated with it, then its nature in terms of a periodic displacement in space and time should be represented by a definite wave equation. In order to obtain this equation, we consider a system of stationary waves to be associated with the particle and referring the particle to the Cartesian coordinate system, at any point x, y, z in the immediate vicinity of the particle, ψ undergoes periodic changes, its value at any instant t being given by

$$\psi = \psi_0 \sin 2\pi\upsilon t \qquad (1)$$

In it, ψ_0 is the amplitude at the point considered, independent of t but function of x, y, z and υ is the frequency.

If the position coordinates of particle be (x, y, z) and ψ be the periodic displacement for the matter waves at any instant t, we can write a differential equation of this wave motion in the classical way as

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{u^2} \cdot \frac{\partial^2\psi}{\partial t^2} = 0 \qquad (2)$$

Where u is the wave velocity of particle, $\psi\,(x, y, z, t)$ usually called the wave function, can be expressed as a periodic displacement in the form of the solution of equation (1) as follows:

$$\psi\,(x, y, z, t) = \psi_0\,(x, y, z)e^{-iwt} \qquad (3)$$

In it, ψ_0 is the amplitude of particle wave at the point (x, y, z). It is independent of time t.

Equation (3) can also be expressed in the terms of position vector $\vec{r}$ given by

$\vec{r} = \hat{\imath}x + \hat{\jmath}\,y + \hat{k}\,z$ As follows:

$$\psi\,(\vec{r}, t) = \psi_0\,(\vec{r})e^{-i\omega t} \qquad (4)$$

If we differentiate equation (4) twice respect to time, we get

$$\frac{\partial^2\psi}{\partial t^2} = i^2\omega^2\psi_0 e^{-i\omega t} = \omega^2\psi_0 e^{-iwt}$$

$$\frac{\partial^2\psi}{\partial t^2} = -\omega^2\,\psi \qquad (5)$$

Substituting the value of $\frac{\partial^2\psi}{\partial t^2}$ from this equation in equation (2), we get

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{\omega^2}{u^2}\psi = 0 \qquad (6)$$

Where $\omega = 2\pi\upsilon\upsilon = 2\pi\frac{u}{\lambda}$.

So that $\frac{\omega}{u} = \frac{2\pi}{\lambda}$ (7)

From equation (6) and (7), we get

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{4\pi^2}{\lambda^2}\psi = 0$$

But $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2, \nabla^2$ being the Laplacian operator

Hence $\nabla^2\psi + \frac{4\pi^2}{\lambda^2}.\psi = 0$ (8)

According to the de Broglie's theory, a particle of mass m moving with velocity v is associated with a wave system of some type of wavelength, $\lambda = \frac{h}{mv}$. Though one has no knowledge of what is that vibrates, one can indicate it by ψ, the periodic changes, which are responsible for $\lambda = \frac{h}{mv}$

Using this relation equation (8) gives

$\nabla^2\psi + \frac{4\pi^2 m^2 v^2}{\hbar^2}.\psi = 0$

$\nabla^2\psi + \frac{m^2 v^2}{\hbar^2}.\psi = 0$ (9) Where $\hbar = \frac{h}{2\pi}$

$\therefore\ \nabla^2\psi + \frac{p^2}{\hbar^2}.\psi = 0$ (Since p, the linear momentum = mv)

The kinetic energy of the particle in terms energy E and Potential energy V can be written as:

$\frac{1}{2}mv^2 = E - V$

Or $mv^2 = 2\ (E-V)$

Or $m^2v^2 = 2m\ (E-V)$

Substituting it in equation (9), we get

$\nabla^2\psi + \frac{2m}{\hbar^2}\ (E - V)\psi = 0$

Or $\nabla^2\psi + \frac{8\pi^2 m}{h^2}\ (E - V)\psi = 0$ (10)

This is the time independent Schrodinger equation in it's usually used from. This equation is also termed as the Schrodinger's fundamental Wave Equation with respect to space. This equation, as it is independent of time gives the steady value. The solutions of this equation are called the steady state wave functions. This equation is particularly useful when the energy of the particle is very small as compared to the rest energy of the particle. In most of the atomic problems, which we consider, the energy of the particle will be very small when compared to the rest energy.

The Schrodinger equation with respect to time can be derived as follows

Schrodinger's Time dependent Wave Equation:

In order to obtain a time dependent Schrodinger equation, we eliminate the total energy E from time independent Schrodinger equation, given by relation

$\nabla^2\psi + \frac{2m}{\hbar^2}\ (E - V)\psi = 0.$ (1)

Since there are no forces acting on a free particle, it can be considered to be having a constant potential energy. For the sake of convenience, this constant potential energy is taken to be zero, i.e., V (x, y, z) = 0. So that equation of motion for a free particle of mass m becomes

$$\nabla^2\psi + \frac{2m}{\hbar^2}E\psi = 0 \qquad (2)$$

For it, let us differentiate equation $\psi\,(x,y,z,t) = \psi_0\,(x,y,z)e^{-iwt}$ with respect to time. It gives

$$\frac{\partial\psi}{\partial t} = -i\omega\psi_0\, e^{-i\omega t} = -i\omega\psi = -2\pi i\upsilon\psi \quad (\text{Since } \omega = 2\pi\upsilon\upsilon)$$

$$\frac{\partial\psi}{\partial t} = -i\,\frac{2\pi E}{h}\psi \qquad \text{Since } E = h\upsilon, \therefore \upsilon = \frac{E}{h}$$

Or $\frac{\partial\psi}{\partial t} = -\frac{iE}{\hbar}\psi$ (3) Since $\hbar = \frac{h}{2\pi}$

Or $E\psi = -\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = i\hbar\,\frac{\partial\psi}{\partial t}$ (4)

Substituting this value in equation (1), we get

$$\nabla^2\psi + \frac{2m}{\hbar^2}\left(i\hbar\frac{\partial\psi}{\partial t} - V\psi\right) = 0$$

or $\frac{\hbar^2}{2m}\,\nabla^2\psi + i\hbar\frac{\partial\psi}{\partial t} - V\psi = 0$

or $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\,\nabla^2\psi + V\psi$

or $i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\,\nabla^2 + V\right)\psi$ (5)

Equation (5) is known as the Schrodinger's time dependent wave equation containing the time factor. It is unique among the differential equations of mathematical physics, as it includes the imaginary quantity $i = \sqrt{-1}$.

The equation can be modified in terms of Hamiltonian operator H given by

$$H = -\frac{\hbar^2}{2m}\,\nabla^2 + V$$

Equation 5) then becomes,

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi$$

$H\psi = E\psi$ (6) using equation 4)

The Hamiltonian H thus coincides with the total energy operator. Although H and E both are the energy operators, yet while H depends only on space coordinates, E depends upon time coordinate.

The Schrodinger equation, like Newton's law of motion, is a fundamental relationship, showing logical coherence to a vast amount of experimental observations. These two equations are like statics and dynamics in classical mechanics; hence derivability of the time dependent equation from the time independent form is much significant [1].

Concluding Remarks:

Schrodinger's equation is useful for investing various quantum mechanical problems. With the help of these equations and boundary conditions, the expression for the wave function is obtained. The probability of finding the particle is calculated by using wave function. Schrodinger's equation is used to solve one dimensional problems in which the potential is discontinuous and is such that between two points of discontinuity it is a constant. e.g. The problems of potential well and potential barrier. This article provides a broad derivation of Schrodinger's wave equations which would helpful for understanding.

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RECENT TRENDS IN DEEP - TECH STARTUPS

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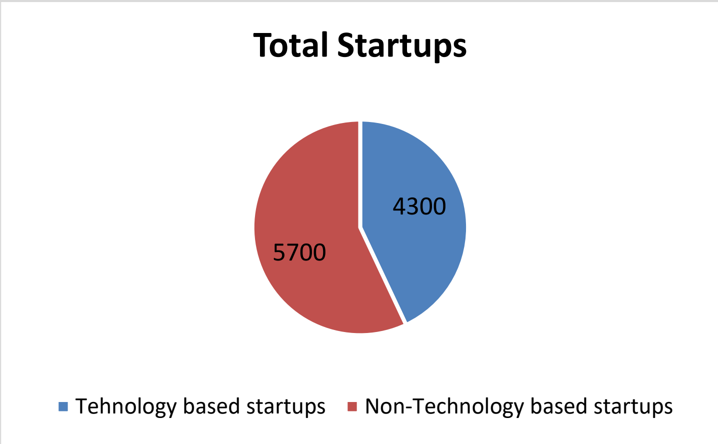
Introduction:

The term Deep Tech was coined in 2014 by Indian investor and entrepreneur Swati Chaturvedi. Deep Tech refers to fundamental breakthroughs in science and engineering that profoundly impact industries and people's live. It brings about a change in the landscape of the affected field such that the change is enduring and permanent. In business and consumer innovations existing technology is used but in Deep Tech companies' revolutionary solutions are created that redefine markets and industry processes. The solutions brought in by the Deep Tech is a paradigm shift in the way things were practiced and goals achieved, such that the resultant ecosystem or product will not have any resemblance to the initial product or current industry practices. For instance Uber, will not be considered as deep tech company because it was built on the concept of a 'sharing economy' and their service was built using existing technologies. Conversely, Deep Tech innovations provides solution to previously-intractable real life problems, for example, data analytics to help farmers grow more food, medical devices and drugs that cure disease and extend life, artificial intelligence to forecast natural disasters such as earthquakes, clean energy solution to lessen the impact of climate change, autonomous vehicles, flying cars, drones.

The Deep Tech ecosystem is greatly benefitted due to the innovations in the input technology required to create the niche product and reduction in cost of the same technology. Concurrently, is an exciting time for India's deep tech startup ecosystem which at a tipping point. According to a report from KPMG, India is the third largest AI startup ecosystem, as 13 percent of the global technology industry leaders indicate its potential for tech breakthroughs. According to grant-thorton 2019 the following is the composition of Indian- startups:

1. The total startups are approximately 10,000 and around 800 technology based new startups are coming up annually.



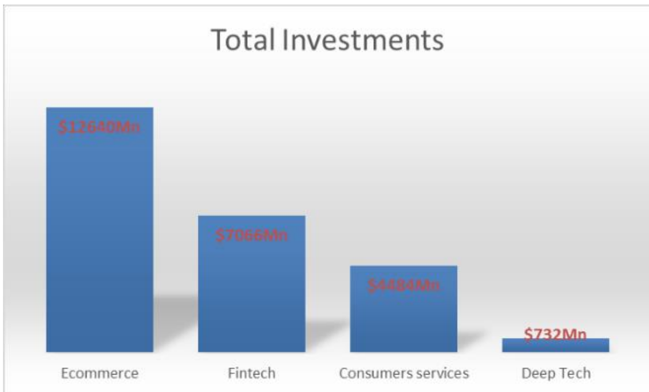
Sector wise concentration:

Sr. No.	Technology based startups	%	Non Technology based startups	%
1	E-commerce	33	Engineering	17
2	B2B	24	Construction	13
3	Consumer internet	12	Agri products	11
4	Mobile apps	10	Textile	8
5	Saas	8	Printing andpackaging	8
6	Others	13	Transportand logistics	6
7			Outsourcing andsupport	5
8			Other	32

The following is the scenario of the deep-tech startups in India.

1. The DeepTech Startups in India received a total investment of $732 Mn in 170 deals between 2014 and 2018.In terms of value, the funding in Deep Tech startups has risen at a CAGR of 22.79% during this period. The numberof deals has increased at a CAGR of 20.11% over the five-year interval.
2. Deep Tech in India is finding it hard to catch up with the top performing sectors of the Indian startup ecosystem.

Bengaluruhas emerged as a hub for the Deep Tech startup as many ancillary systems are in place for this eco system to develop; Hyderabad also benefits from the advantages of the former, but is developing at a slower pace. Being in the nascent stage and having greater appeal and visibility Deep Tech companies have great life expectancy and low rate of failure. But the Deep Tech ecosystem is still developing, with the fundamentals of the industry still not defined and supply chain yet to firm up, the funding in deep-tech is relatively lower compared to the top funded and established sectors. High setup cost, weak intellectual property framework, and low market adoption are some of the reasons for weak investor interest in the India Deep Tech startups.



Objective:

We consider emerging deep technology, and its potential as startup and challenges in India.

Research Methodology:

The study is mainly based on the secondary data. The data are collected from journals, reports, web, magazines, data reports etc. The study is descriptive and conceptual in nature.

Specificities of Deep Tech Startups:

Deep Tech start-ups transfer research andtechnology to the market which due to their high worth and use of new age technology, results in high impact for society. Deep Tech innovations have lowered the costs of starting a company and created an ecosystem, in which many companies benefit due to lateral use and application of products and processes. Having stated as such, still it is slower and more expensive to establish an innovative Deep Tech idea and get valuable derivatives or products from such ideas, which can be used in the industry or society. The major challenges and shortcomings to get the Deep Tech companies' off the ground as compared to a digital startup for the following reasons:

1. Strong research and innovations:

Deep Tech startups, by their very nature are using the best technology and methodology for making products or processes. Thus, they have a strong foundation in cutting edge research and develop new paradigms. They have to create a niche product or conceive a new approach, which requires a synergy of manpower and attendant technology for getting the desired results. Some of the challenges towards intensive research and innovations are:

A. **Advanced skills:** Human resource required for Deep Tech companies have to be the most creative and those with an ability to bring forth 'out of box' solutions. These individuals will have to come from multitude fields and their leaders will have to bring the necessary synergy for meaningful and positive outcomes.
B. **Knowledge:** Coming from the previous point, the collective knowledge base required for setting up a Deep Tech company will be very challenging. Since, the tools employed to come at a result are also the very niche that the industry can provide. Therefore, subject experts will be required ranging from software developers, artificial intelligence experts, machine experts, production specialist etc. to bring in meaning outcomes.
C. **Infrastructure:** Since most of the tools and inputs being worked with in a Deep Tech company would be used in the scale and scope for the first time, the infrastructure and initial cost will be very high. The unique requirements of the work station will also entail additional cost to the promoters.
D. **Lengthen the product time in the market:** The Deep Tech products at an early stage will be for high tech solutions and gradually volumes will bring the cost down, thus scaling up will also entail extra cost due to large gestation periods. Hence, the profitability of the venture will take longer durations to realize.

2. Large funds requirements:

Given the unique nature of Deep Tech companies, many of the reasons listed above, the initial capital required will be very high. Also, most of the sourcing of tools and inputs required, might not be available locally, which would increase the cost many folds. The outsourcing of certain portions to experts in other fields will also increase the costs.

3. Heavy Industrialization process:

If the Deep Tech product has to be set up in a industrial framework, then there would be a need for total revamp of the process or setting up from ground up. Even if the

Deep Tech Company is small in scope and scale, the technology input, hardware requirement, networking and related inputs will be much higher than the industry standards.

4. Yet to defined commercial applications:

Many of the products and processes developed by the Deep Tech companies might not find use in immediate future, due to lack of other infrastructure and conditions. Thus some of the innovations might not be monetized immediately, which will generate negative perceptions towards the Deep Tech startups. For example,blockchainwas used by the developers to create a product, namely Bitcoin. But the same technology is being used for many products now, but the developers of blockchain did not foresee the huge potential of their innovations but they opened the door for a new market in finance pioneered by Bitcoin.

Deep Tech Innovation Ecosystem:

Research has established that innovation depends on smoothly functioning innovation ecosystems—combinations of people, companies, infrastructure, and government policies linked through informal and formal networks. Collaborations are crucial to the concept of Open Innovation, which—unlike closed, in-house innovation processes—requires that companies' source and collaborate with others. These collaborations is not limited in scope or the size of collaborating institutions or entities. The most notable rising trend, however, is the growing role of startups and their increasingly important collaborations with corporates. Startups today bet on long-term and high-risk deep-tech innovation—such as storage of information in DNA, or antimatter propulsion—that used to be the prerogative of public research. Collaborations between startups and corporates began in ICT and biopharma but are now spreading through all industries.

The role of startups in the innovation ecosystem is increasing, especially within the field of radical innovation. Collaborations between corporates that tend to implement incremental innovation and academic centers that focus on radical innovation can be cumbersome due to strategic misalignments. Still, the synergy is important to bring for an equitable system such that the scope of work of academic centers and corporates are narrowed.

Collaborative ecosystems and open innovation are crucial in Deep Tech. This is due to the fore mentioned specifics of Deep Tech innovation, which pose daunting challenges for a young company.

Important components of Deep Tech Startups:

The four key dimensions need to be connected from the beginning of the creation process of Deep Tech start-ups are:

1. Minds:

The human resource required for a promising Deep Tech startup would involve experts from various fields, mostly from technology related fields, including software, hardware, biotech, artificial intelligence, robotics to name a few. These people will also require access to the best technological infrastructure and inputs to bring an idea to fruition. These innovators will bring a promising marketing oriented technology into a product or process with strong intellectual property, leading to reorientation in the present concepts or creation of new markets.

2. Management:

The team leading the Deep Tech startup will require strong ideas, smart implementation and a versatile team of experts across many fields. The entrepreneur/ intrapreneur will have to have a good market vision, who is able to convert technology led disruptive technologies into ready to invest business opportunities or products that can be readily monetized.

3. Market:

These Deep Tech products will have to be marketed very strategically to make them viable in the industry. The same will require interested partners and committed industrial clients, who have the acumen to bring see the development and distribution of these products.

4**. Money**:

The capital for Deep Tech will not be available from traditional sources and will require investors with deep understanding of technology and ways in which they can be transformed into good business value. The investor will require multiple ventures, such that some can become successful thereby balancing the investment.

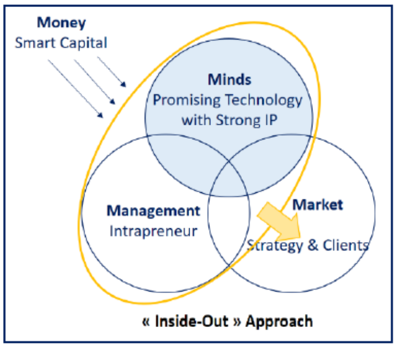
Incentives must be aligned between these four dimensions to ensure compatibility and success. Most of the time, a strong and smart team of Minds and Management developing a promising technology with a smart Market strategy including several committed industrial clients will make it much easier to attract and secure smart Money.

Two Models of Startups:

Based on the above key components we have two models of Deep Tech startups: either create a new startup or to help the existing startups to access technology.

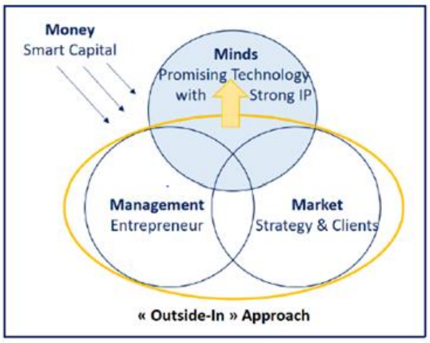
1. Inside-Out Approach:

This approach is adopted when the new technology cannot be transferred to existing companies. RandD adopts a market-oriented approach at early stage of development with the objective to develop strong IP and transfer the technology to the market. This approach is a symbiosis of a smart entrepreneur coming together with experts in the field and innovators in the industry to make marketable products. The focus here is on innovating new products and adapting them to the market requirement.



2. Outside-in Approach:

In this approach the entrepreneurs with early stage of startups and having good vision of the market and potential clients contact the RandD with promising technology portfolio for the new business. The focus here is on the marketability of the products with the experts being motivated to come up mass appeal products or the specific needs of industry which can be monetized far easier.



Key Challenges of Startups, Indian Perspective:

The following are some of the challenges faced by Deep Tech startups:

1. Time to Marketing:

The Deep Tech startups are based on new technology and so their products may take longer development time. It takes on an average 4 years to develop a technology in biotech (1.8 years from incorporation to first prototype and another 2.2 years to reach the market) while about 2.4 years (1.4 years to first prototype and 1 year to market) for startup in blockchain as shown in the analysis of the startups done by Hello Tomorrow Challenge. In India, with the lack of specialized 'venture capitalists', who have deep knowledge of technology, the chances of an entrepreneur waiting for long gestation period for profitability is very low.

2. Capital Intensity:

The understanding and maturity in investment is still very low in India, thus there is a general lack of focused credit disbursement for niche innovation and technologies. For developing new products expensive infrastructure is required. Also there is extended period for the product. Hardware is more expensive then software. For example to develop a mobile app is not expensive as compared to designing a machine say, for hearing aid.

3. Technology risk and complexity:

The lack of incubators for the experimental technology to mature is also great impediment towards the development of the tech ecosystem.Deep Tech ecosystem is nascent and emerging and many aspects like supply chains are not stabilized. They are difficult to navigate as it requires thorough analysis of the stakeholder's interdependence and value creation models. Due to its complexity and risk factors newcomers finds themselves in unfamiliar territory.

4. Lack of high-skilled workers:

A specialized workforce is required for innovations in advance technologies such as robotics, AI, drones. Though India have the required talents but most of them prefer to work in the western countries for better pay prospects. This migration has negative impact on Deep Tech.

5. A Low-Level Readiness of the Indian Public towards technology:

India is leading exporter in software services but Indians are less tech-savvy compared to other nations. This makes it difficult for companies to capture a larger share of the 1.2 Bn-strong vibrant Indian market.

6. Regulatory frame work:

India has weak intellectual property framework.Although the Modi led BJP government has taken many steps towards the intellectual property rights, but there is still lack of confidence of innovators.

Current trends in Deep Tech startups in India:

In today's era of Deep Tech, the startup ecosystem is playing a key role in its development. In 2018, Deep Tech saw a rise of 20% in private investment since 2015 and has reached $18 billion globally. The following are the current key trends which are being accelerated by promising startups in this field.

1. Artificial Intelligence (AI) in healthcare:

In healthcare sectors Deep Tech startups are making steady and much needed innovations. For instance, Audiologist couple T Uday Raga Kiran and Remyafounded Nautilus Hearingthe Hubli-based startup, incubated by Sandbox Hubli. They have designed a device that can be used to conduct ear tests with ease.The device costs Rs 2 lakh, claiming the price point is 80 percent lower than that of other devices available in the market at present.Niramai, another startup has created technology that diagnoses breast cancer by integrating AI with Machine Learning.

2. Virtual Reality (VR) in real estate and collaborative platform:

The real estate is benefitted immensely by the developments in VR technology. Now homebuyers can have the virtual view of their prospective new homes as compared to traditional 2- dimensional layouts. Magicbricks- Mumbai based VR startup have established the India's first real estate experience center in the city. Meraki Studio, another Mumbai based VR startup enlisted by the Lodha group to create virtual walkthroughs of its properties in, an upcoming smart city in Palava, near Mumbai.

VR is also set to revolutionize collaboration. Hyderabad –based startup NuSpace founded by Hemanth Satyanarayana is based on the idea of interacting between the people through holograms instead of being physically present in the same room together. Financial company MetLife is already using NuSpace to enhance its customer experience by enabling customers to talk to 'avatars' of customer service agents.

3. Internet of things (IoT) in sustainability:

IoT has seen a rise since 2015 in the Indian startup ecosystem. IoT funding have raised to $54.5 million (approx..) by May 2018 and it is predicted by NASSCOM and Deloitte report in 2018 that India will soon have 1.9 billion IoT devices installed with $9 billion market value. There are wide range of fields that uses IoT such as automobiles and logistics. The main purpose of IoT startups is to reduce energy wastage. For instance, Bengaluru-based Zenatix is curbing energy wastage for commercial establishments byusing machine-learning based models. Another startup on the same lines is Ahmedabad-based Ecolibrium Energy.

4. Robotics:

Robotics is another fast growing field of Deep Tech startups. It is widely used in retail, military, e-commerce, logistics, hospitality and also in agriculture. Maruti Drones a robotics startup designs Unmanned Aerial Vehicle (UAV) commonly known as drones. These agri-sparying drones not only effectively monitor but also provide data about crop and soil health, hence boosting food production. Another Chennai based restaurant Robot is first one to design Robot waiter in India.Generobotics- Thiruvananthapurm based startup launched in 2015 by engineers VimalGovind MK, Arun George, Nikhil NP, and Rashid Bin Abdulla Khan has designed a spider-shaped robot – Bandicoot to clean holes.

5. Computer vision is aiding retail, healthcare and transport:

Computer vision combines the elements of data processing, analytics, machine learning and AI. This technology enables computers to process images and has several applications,including e-commerce, transport, security and health care.Bangaluru based Hyperworks uses AI in the perishable food market. They provide self-checkout counter that can scan food items using computer vision and not barcodes, in turn make business easier for merchants. Another Bengaluru based startup Turing Analytics helps shoppers to look for products using images and videos. This has eased the business between retailers and consumers. Another startup, Netradyne,enables safe driving by alerting drivers to any emergency on their route using computer vision technology , hence aiding the transport industry.

Conclusion:

The new Deep Tech ecosystem is in early stage and is maturing as an industry in the fields of AI, robotics, blockchain, agritech, VR.The rise of new platform technologies, falling barriers and the inevitability of climbing the technology ladder are the major driving factors. The third factor is the significant decrease of barriers to entry into cutting-edge research, including more and open data (open IP environment, etc.), hardware (cost decrease, 3D printing and IoT), software and computing (AI and computing power), biology as well as process (interdisciplinary research and open innovation practices).As new discoveries are being made and technologies demonstrate their potentials various players, their role and rules will eventually evolve. Having seen the potential, it is also important to understand the unique needs of the industry, challenges in capital and manpower, markets for the products and navigating the regulations, many of which are in the drafting stage. Still, the stake-holders should set their goals as first rule and get into the game. They should be ready to take risk and should be prepared to learn including from their failures, and only then they can tap into the power and potential of the win-win ecosystem.

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VARIOUS METHODS OF APPROXIMATION IN NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract:

Mathematical formulation, results of many physical problems in differential equations which are truly nonlinear. In several cases it is possible to replace such a nonlinear equation by a related linear equation which approximates the actual nonlinear equation closely enough to give useful results. However such a linearization is not always possible, and when it is not, the original nonlinear equation itself must be considered.

The study of nonlinear equation is generally limited to a variety of rather special cases, and one must option to various methods of approximation. Here we shall give a brief introduction to certain of these methods.

Keywords: Nonlinear, Approximation, Differential Equation, Phase Plane, Stability.

Introduction:

The laws of the universe are written in the language of mathematics. Algebra is sufficient to solve several static problems, but the most interesting natural phenomena involve change and are described by equations that relate changing quantities.

When we study the differential equations, it has three principal goals, first is to discover the differential equation that describes a specified physical situation, second is to find either exactly or approximately the appropriate solution of that equation and lastly, to interpret the solution that is found.

Linear equations are the most tractable, they are the ones that we understand best, and they are the ones for which there is the most complete theory. But in fact the world is largely nonlinear. We have traditionally shied away from nonlinear equations just because they are so difficult, and because their solutions can rarely be written down in a closed formula. But there is still much that can be said about nonlinear differential equations, especially if we are willing to accept qualitative information rather than closed formulas, there is a considerable amount that one can learn using even elementary techniques.

In the present paper we shall concentrate one attention on nonlinear equations of the form $\frac{d^2x}{dt^2} = f(x, \frac{dx}{dt})$ …….. (1)

is the resulting equation of motion.

In equation (1), $f(x, \frac{dx}{dt})$ is the force acting on it, the values of x is the position and $\frac{dx}{dt}$ is the velocity, i.e. the qualities that at each instant characterize the state of the system, the phases of the system, the plane determined by these two variables is called the phase plane.

By substituting $y = \frac{dx}{dt}$ and t as a parameter and a curve in the $x - y$ plane i.e. the phase plane are continuously differentiable, in which the dependent variable t does not appear in the functions on the right is called an autonomous system.

The study of a differential equation can be divided in three stages:

i. Formation of differential equation, from the given physical problem.
ii. To get the solution of the differential equation, and fixing the values of the arbitrary constants, with the help of given conditions.
iii. Physical analysis of the mathematical solution.

The Phase Plane and its Phenomena:

i. Phase Portraits

If the initial point (x_0, y_0) is not a critical point, then the corresponding trajectory is a curve in the xy-plane along which the point $(x(s), (y(s))$ moves as s increases. Here we can exhibit qualitatively the behavior of solutions of the autonomous system $\frac{dx}{dt} = P(x, y)$, $\frac{dy}{dt}$ $Q(x, y)$ by constructing a picture that shows its critical points together with a collection of typical solution curves or trajectories in the xy-plane. Such a picture is called a phase portrait or phase plane picture because it illustrates phases or xy-states of the system, and indicates how they change with time.

e.g.$\frac{dx}{dt} = x^2 - 2xy$,$\frac{dx}{dt} = 2xy - y^2$.

Consider the first order differential equation of the form

$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}$, which may be difficult or impossible to solve explicitly. Its solution curves can nevertheless be plotted as trajectories of the corresponding autonomous two-dimensional system $\frac{dx}{dt} = P(x, y)$, $\frac{dy}{dt}$ $Q(x, y)$.

Paths:

If f and g are not both constant functions, then the equations $x = f(s)$ and $y = g(s)$ defines a curve in thexy-plane which we shall call a path for orbit or trajectory of the system $\frac{dx}{dt} = P(x, y)$, $\frac{dy}{dt}\ Q(x, y)$.

Stability:

A point (x_*, y_*) of the autonomous system in $\frac{dx}{dt} = P(x, y)$, $\frac{dy}{dt}\ Q(x, y)$ is said to be stable provided that if the initial point (x_0, y_0) is sufficiently close to (x_*, y_*) then $(x(s), (y(s))$ remains close to (x_*, y_*) for all$s > 0$.

Asymptotic Stability:

The critical point (x_*, y_*) is called asymptotically stable if it is stable and moreover every trajectory that begins sufficiently close to (x_*, y_*) also approaches (x_*, y_*)as $s \to +\infty$.

Types of Critical Points:

Definition:

The isolated critical point (0,0) of the equations of the autonomous system $\frac{dx}{dt} = P(x, y)$, $\frac{dy}{dt}\ Q(x, y)$ is called a center if there exists a neighborhood of (0,0) which contains a countably infinite number of closed paths P_n $(n = 1,2,3, \ldots.)$, each of which contains (0,0) in its interior, and which are such that the diameters of the paths approach 0 as $n \to \infty$ (but (0,0) is not approached by any path either as $s \to +\infty$ or as $s \to -\infty$). [2]

i. Nodes:

A node is said to be proper provided that no two different pairs of opposite trajectories are tangent to the same straight line through the critical points. [3]

The critical point (x_*, y_*) of the autonomous system in the equation $\frac{dx}{dt} = P(x, y)$, $\frac{dy}{dt}\ Q(x, y)$ is called a node provided that either every trajectory approaches (x_*, y_*) as $s \to +\infty$ and every trajectory is tangent at (x_*, y_*) to some straight line through the critical point.

ii. Saddle Points:

A critical Point is approached and entered by two half line paths as $s \to +\infty$ and these two paths lie on a single line.It is also approached another two half line paths on another line. Between the four half line paths there are four regions and each contains a family of paths resembling hyperbolas.

iii. Centers:

A Center also called as vortex is a critical point that is surrounded by a family of closed paths. It is not approached by any path as $s \to +\infty$ or as $s \to -\infty$. [1]

iv. Spirals:

A critical Point is approached in a spiral like manner by a family of paths that wind around it an infinite number of times as $s \to +\infty$or$s \to -\infty$.

Conclusion:

Differential equations are applicable in various branches of Mathematics, Physics and Engineering, etc. In Dynamics whenever any body is found in motion, it certainly retains some differential equations. e.g. Population, electric circuits, proportion, temperature, etc., to solve explicitly, its solution curves can nonetheless be plotted as trajectories of the corresponding autonomous two-dimensional system. For each of the autonomous system we find the real critical points of the system and obtain the differential equation which gives the slope of the tangent to the paths of the system and to solve this differential equation we obtain the one parameter family of paths.

These methods of approximations are useful to solve explicitly the differential equations that are difficult or impossible to solve.

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THE ACHROMATIC NUMBER OF SPLITTING GRAPHS

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Abstract:

Let $S(G)$, the splitting graph of a graph G which can be obtained by taking a new point v' for each point v of G and joining v' to all points of G adjacent to v. In this study, the achromatic colouring of the splitting graphs of double star graph, path, star graph, comb graph and complete graph are discussed. Along with this, the investigation of the properties of these graphs is also done.

Keywords: Achromatic number, Splitting graphs, Star graph, Comb graph, complete graph

AMS Subject Classification: 05C15

Introduction:

The splitting graph was introduced by E. Sampath kumar and B. Walikar in the year 1981. In that paper they introduced the definition of the splitting graph, studied some of its properties and obtained its characterization. For each node v of a graph G, take a new node v'. Join v' to all the nodes of G adjacent to v. The graph $S(G)$ thus obtained is called the splitting graph of G.

In a graph G, the achromatic colouring refers to a proper vertex colouring in a manner that there is at least one line incident on every colour pair. For a graph G, the maximum number of colours possible in such a colouring of G, is its achromatic number, denoted by $\psi(G)$. A Star a_n is the bi partite graph $K_{1,n}$. A double star graph is the graph $K_2(a_n, a_r)$ obtained by joining the root nodes of two star graphs a_n and a_r by an edge. Comb is the graph got by joining a single pendent edge to each point of a path.

The Achromatic Number of Splitting graph of Path graph:

Structural Properties of $S(P_n)$:

- The total number of points in $(P_n) = 2n$.
- The total number of lines in $S(P_n) = 3n - 3$.
- The maximum degree in $S(P_n) = 4$.
- The minimum degree in $S(P_n) = 1$.

Observation:

- For any $S(P_n)$, the achromatic number

$$\psi[S(P_n)] = \begin{cases} 3, & n = 2,3 \\ 4, & n = 4 \\ 5, & n = 5,6 \\ 6, & n = 7,8 \\ 7, & n = 9,10,11 \\ 8, & n = 12,13 \\ 9, & n = 14,15 \end{cases}$$

The Achromatic Number of Splitting graph of Star graph

Structural Propertiesof $S(K_{1,n})$

- The total number of points in $S(K_{1,n}) = 2n$.
- The total number of lines in $S(K_{1,n}) = 3n$.
- The maximum degree in $S(K_{1,n}) = 2n$.
- The minimum degree in $S(K_{1,n}) = 1$.

Theorem:

For any $S(K_{1,n})$, the achromatic number $\psi[S(K_{1,n})] = 3$ for $n \geq 2$.

Proof:

Let $\{v_0, v_1, \dots, v_n\}$ be the vertex set of $K_{1,n}$where v_0 denotes the root node. Assign the remaining nodes to the leaf nodes of $K_{1,n}$. By the definition of $S(K_{1,n})$, let u_0 be the corresponding image of v_0 and $u_1, \dots, u_n$be the nodes corresponding to $v_1, \dots, v_n$. Here the node u_0 is adjacent to all the other u_i's and v_0 is adjacent to the other v_i's. Let us take set of colours $C' = \{C_0', C_1', C_2'\}$. Assign C_0 to v_0 and C_1' to all the v_i's, for $1 \leq i \leq n$ and C_2' to all the u_i 's for $1 \le i \le n$. In $S(K_{1,n})$, only two nodes have degree greater than 2. We cannot assign more than 3 colours. By this construction, the colouring is achromatic and it is the maximal one.

Example:

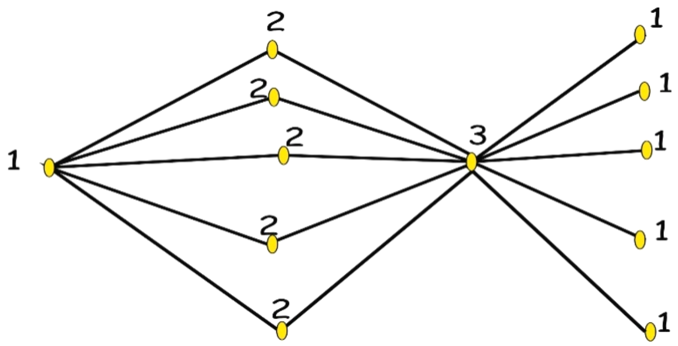

Figure1: $\boldsymbol{\psi[S(K_{1,5})] = 3}$

The Structural properties of $S(K_2(a_n, a_r))$:

- The total number of points in $K_2(a_n, a_r) = n + r + 2$.
- The total number of lines in $K_2(a_n, a_r) = n + r + 1$.
- The total number of points in $S(K_2(a_n, a_r)) = 2(n + r + 2)$.
- The total number of edges in $S(K_2(a_n, a_r)) = 3(n + r + 1)$.

Observation:

For any $S(K_2(a_n, a_r))$,the achromatic number $\psi[S(K_2(a_n, a_r))] = 5$

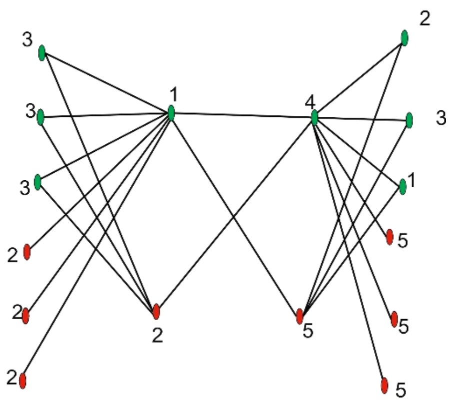

Figure 2: $\boldsymbol{\psi[S(K_2(a_3, a_3))] = 5}$

The Achromatic Number of Splitting graph of complete graph and Comb graph

Structural Properties of $S(K_n)$:

- The total number of points in $S(K_n) = 2n$.

- The total number of lines in $S(K_n) = \frac{3n(n-1)}{2}$.
- The maximum degree in $S(K_n) = 2(n-1)$.
- The minimum degree in $S(K_n) = n-1$.

Theorem:

For any $S(K_n)$, the achromatic number $\psi[S(K_n)] = n+1,\ n \geq 3$.

Proof:

Let K_n be the complete graph. The splitting graph $S(K_n)$ is obtained by taking a new point u_i for each point v_i of K_n and joining u_i to all points of K_n adjacent to v_i. Consider the two sets of vertices $U = \{u_1, u_2, \dots u_n\}$ and $V = \{v_1, v_2, \dots v_n\}$. Assign the nodes in V to K_n in the positive direction, the anti-clock wise direction is considered as a positive direction. Assign the nodes in U to the newly introduced nodes in $S(K_n)$ in the same direction.

By the definition of complete graph all the v_i's are adjacent to all the other $n-1$ nodes. We need n colours to colour K_n. In $S(K_n)$, the nodes u_i's are regular with degree $n-1$. Here all the u_i's are adjacent to v_j's for all $j \neq i$. Consider the colour set $C' = \{C_0', C_1', \dots, C_n'\}$. Allot C_i' to v_i for $1 \leq i \leq n$ and C_0' to all the u_i's. By this construction the above colouring is achromatic and is the maximal one. Hence $\psi[S(K_n)] = n+1,\ n \geq 3$.

Example:

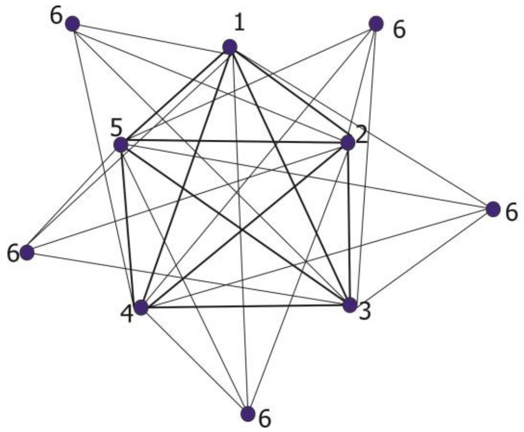

Figure 3: $\psi[S(K_5)] = 6, n \geq 3$.

Structural Properties of $S[Cb_n]$:

- The total number of lines in $S[Cb_n] = 6n-3$.
- The maximum degree in $S[Cb_n] = 6$.
- The total number of points in $S[Cb_n] = 2n$.

Observation:

- For any $S(Cb_n)$, the achromatic number $\psi[S(Cb_n)] = n + 2,\ n \geq 2$.

Conclusion:

In this work, the structural properties of the splitting graphs of path graphs, star related graphs, complete graphs and comb graphs have been studied.

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THE b-CHROMATIC NUMBER OF SPLITTING GRAPHS

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Abstract:

Let $S(G)$, the splitting graph of a graph G which can be obtained by taking a new point v' for each point v of G and joining v' to all points of G adjacent to v. In this study, the b-chromatic colouring, of the splitting graphs of double star graph, path, star graph, comb graph and complete graph are discussed.

Keywords: b-chromatic number, Splitting graphs, Star graph, Comb graph, complete graph

AMS Subject Classification: 05C15

Introduction:

The splitting graph was introduced by E. Sampath Kumar and B. Walikar in the year 1981. In that paper they introduced the definition of the splitting graph, studied some of its properties and obtained its characterization. For each node v of a graph G, take a new node v'. Join v' to all the nodes of G adjacent to v. The graph $S(G)$ thus obtained is called the splitting graph of G.

The concept of b-chromatic number was introduced by Irwing and Manlove. They defined the b-chromatic colouring as "The b-chromatic number $\varphi(G)$ of a graph G is the largest integer k, such that G admits a proper $k-$ colouring, and every colour class has a representative point adjacent at least to one point in each other class. " This type of colouring is called b-colouring. A Star a_n is the bi partite graph $K_{1,n}$. A double star graph is the graph $K_2(a_n, a_r)$ obtained by joining the root nodes of two star graphs a_n and a_r by an edge. Comb is the graph got by joining a single pendent edge to each point of a path.

The b-chromatic Number of Splitting graph of Path graph:

Observation:

For any $S(P_n)$, the b-chromatic number

$$\varphi[S(P_n)] = \begin{cases} 2,\ n = 2,3,4 \\ 3,\ n = 5 \\ 4, n = 6,7 \\ 5, n \geq 8 \end{cases}$$

Example:

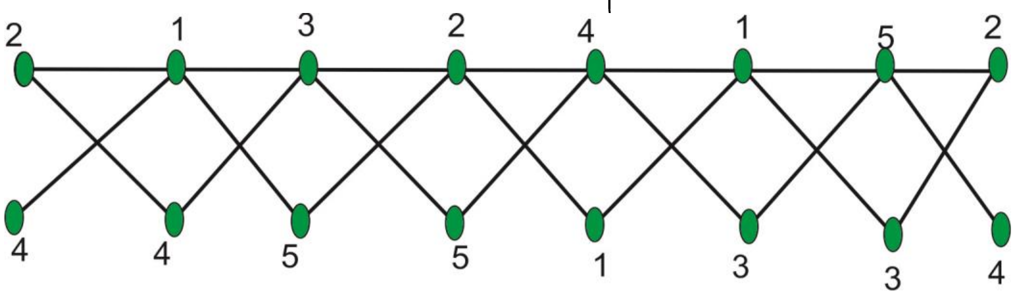

Figure 1: $\boldsymbol{\varphi[S(P_8)] = 5}$

The b-chromatic Number of Splitting graph of Star graph:

Observation:

- For any $S(K_{1,n})$, the b-chromatic number $\varphi[S(K_{1,n})] = 2, n \geq 2$

Example

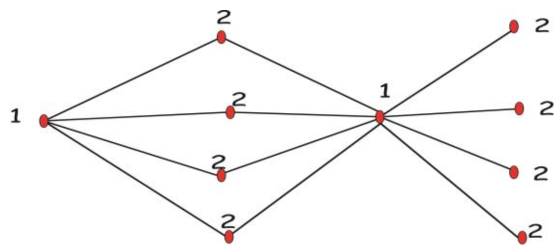

Figure 2: $\boldsymbol{\varphi[S(K_{1,4})] = 2, n \geq 2}$

Theorem:

For any $S(K_2(a_n, a_r))$, the b-chromatic number $\varphi[S(K_2(a_n, a_r))] = 2$

Proof:

Let $K_2(a_n, a_r)$ denote the double star graph which is obtained by connecting the roots of two star graphs a_n and a_r *by* an edge. Let the nodes of $K_{1,n}$ be $\{v_0, v_1, .., v_n\}$ where v_0 is the root node and let the nodes of a_r be $\{u_0, u_1, .., u_r\}$ where u_0 is the root node. The graph $S[K_2(a_n, a_r)]$ can be obtained by adding new nodes $v_i^{'}$ and $u_i^{'}$ for each point v_i and u_i of G and joining $v_i^{'}$ and $u_i^{'}$ to all points of $K_2(a_n, a_r)$ adjacent to v_i & u_i respectively.

The degree of the node v_0 is equal to $n+1$ and the degree of u_0 is equal to $r+1$. Consider the colour set $C = \{C_1, C_2\}$. Assign the colour C_1 to u_0 and C_2 to v_0. Assign the colour C_2 to all the adjacent nodes of u_0 and assign C_1 to all the adjacent nodes of v_0. If we assign any new colour C_3 to any of the $u_i^{'} s$ or $v_i^{'} s$, that will not satisfy the b-chromatic colouring property. Hence the maximal number of colours is 2 and this colouring is b-chromatic. Hence $\varphi[S(K_2(a_n, a_r))] = 2$.

Example:

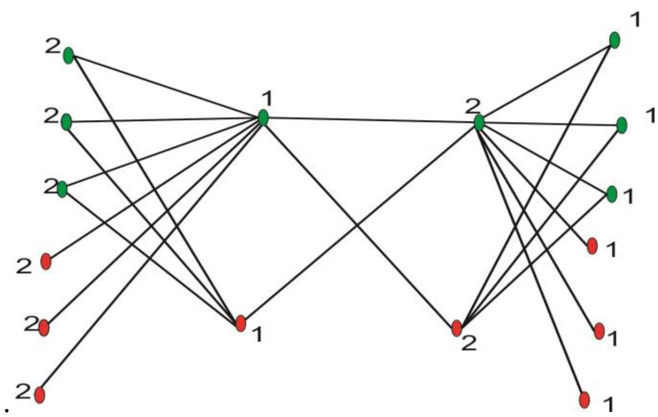

Figure 3: $\varphi[S(K_2(a_n, a_r))] = 2$

The b-chromatic Number of Splitting graph of complete graph and Comb graph

Theorem:

For any $S(K_n)$, the b-chromatic number $\varphi[S(K_n)] = n,\ n \geq 3$.

Proof:

Let $V(K_n) = \{v_1, v_2, \ldots v_n\}$ and $V[S(K_n)] = \{u_1, u_2, \ldots u_n\} \cup \{v_1, v_2, \ldots v_n\}$ where u_i is the node in the splitting graph corresponding to the nodev_i, $1 \leq i \leq n$. Here it is observed that, the node v_i, $1 \leq i \leq n$ has degree $2(n-1)$. All the node su_i have degree $(n-1)$. For $1 \leq i \leq n$, assign $C_i^{'}$ to v_i, we note that n colours are needed to colour these $2(n-1)$ nodes. If we assign a number $C_0^{'}$ to any node u_i, that will violate the property of b-chromatic colouring. Hence n number of colours are required to colour $S(K_n)$. Hence $\varphi[S(K_n)] = n,\ n \geq 3$.

Observation:

For any $S(Cb_n)$, the b-chromatic number $$\varphi[S(Cb_n)] = \begin{cases} 3, n = 2,3 \\ 4, n = 4,5 \\ 5, n = 6,7 \\ 6, n = 8,9 \\ 7, n \geq 10 \end{cases}$$

Example:

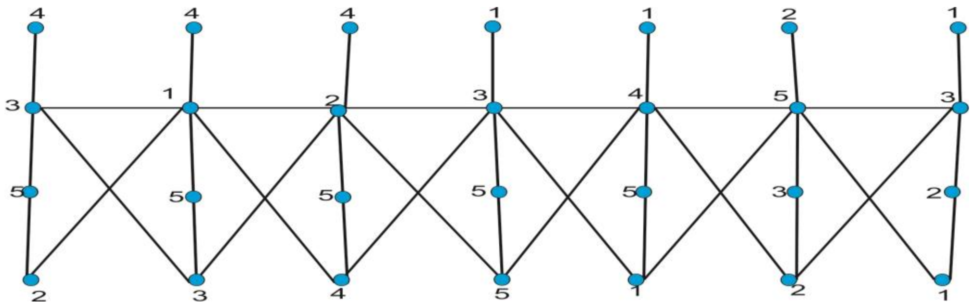

Figure 4: $\varphi[S(Cb_7)] = 5$

Conclusion:

In this work, the structural properties of the splitting graphs of path graphs, star related graphs, complete graphs and comb graphs have been studied.

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STRONG SELF FUNCTION CHAINABILITY BETWEEN TWO SETS IN BITOPOLOGICAL SPACES

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Abstract:

In this work, the notion of strong-self-function chainable sets has been pulled-out to bitopological spaces, which has been defined for general topological space in Shrivastava *et al.* (2014). Results demonstrated by Shrivastava, *et al.* (2014) are drawn-out to strong-self-function chainable sets for a bitopological space. In this article, self$-bi-f-\epsilon-$chainable and strongly self$-bi-f-$ chainable are discussed for bitoplogical space and some results related to them are established in addition to the results established in Vijeta (2020)

Subject Classification: AMS (2000):54A99

Keywords: $f-\epsilon-$chainablility, function – bi $-f-$chainable sets, function– bi–f $-$chainable space

All through this article, X will symbolize a bitopological space with topologies τ_1 and τ_2 and $\tau = \tau_1 \cap \tau_2$ and $f : (X, \tau) \rightarrow [0, \infty)$ will be talked about as a non-constant real valued continuous function unless otherwise stated.

Introduction:

As long back as in 1883, Cantor defined connectedness in metric spaces through ε -chains which have been studied extensively by many mathematicians. These ε-chains were defined between two given points and comprised points of the metric space under consideration. In the context of metric spaces, ε- chainability describes connected sets amongst compact sets. In the same setting, Beer (1981) has characterized compact sets among the connected sets. Different definitions of chains in metric spaces have been provided by various Mathematicians. A simple chain joining two points p, q of the metric space X is a finite sequence $A_1, A_2, \dots, A_m$ of subsets of X such that A_1 (and only A_1) contains p, A_m (and only A_m) contains q and $A_i \cap A_j = \emptyset$ if and only if $|i-j| > 1$. However, Piotr Minc (1990) has called such a chain by the name weak chain, whereas Newmann (1961) has called it a chain of sets. The idea of ε-chainability in metric spaces have been generalized to topological spaces utilizing open covers and have acquired a characterization of connectedness in terms of open cover by Naimpally and Pareek (pre-print). The concept of connectedness by uniformly short paths was defined by Bellamy (1987). Shrivastava and Agrawal (2002) have defined chainable sets using chains between points. These concepts were extended to topological spaces through continuous functions by Vijeta *et al.* (2013). Function $-\varepsilon-$ chainability has been defined in terms of real valued continuous functions on the space. Shrivastava *et al.* (2014) introduced the concept of self and strongly function chainable sets in topological spaces. In this article, the same concept introduced by Shrivastava *et al.* (2014) has been extended to bitopological spaces.

All the way through this article, $[0, \infty)$ is with usual metric topology and ε is positive real number unless stated otherwise. Also τ is the intersection of two topologies τ_1 and τ_2 on X. Consider a bitopological space (X, τ_1, τ_2) and $A \subset X$ then $(A, \tau_{1A}, \tau_{2A})$ is a bitopological space which is subspace of (X, τ_1, τ_2) where (X, τ) is a topological space and τ_{1A} and τ_{2A} are relative topologies on the subset A.

Preliminary:

Function$-bi-f-\epsilon-$ chainable:

The following definitions are defined in Vijeta *et al.* (2018) and Vijeta and Saratha (2018).

Definition 1:

(X, τ_1, τ_2), a bitopological space on set X is termed as function$-bi-f-\epsilon-$ chainable if for $\epsilon > 0$ there is a non-constant function $f: (X, \tau) \to [0, \infty)$, where $\tau = \tau_1 \cap \tau_2$ such that it is continuous and for every two points $x, y \in X$, a sequence $x = x_0, x_1, x_2, \dots, x_n = y$ of points exists in X satisfying the condition $|f(x_i) - f(x_{i-1})| < \epsilon$; $1 \le i \le n$.

Definition 2:

Consider (X, τ_1, τ_2) as a bitopological space and assume that a continuous map f from $(X, \tau_1 \cap \tau_2)$ to $[0, \infty)$ exists that is at the same time non-constant and satisfies the condition that for each positive ε, the space X is function $-$ bi $-f-\epsilon-$chainable. Then the bitopological space (X, τ_1, τ_2) is termed as function $-$ bi $-f-$chainable.

Definition 3:

Consider A *and* B to be the subsets of X. A function $-$ bi–$f - \epsilon$ –chain of length n from set A to set B is a finite collection of subsets of X say $A_{0,}, A_1, A_2, \dots, A_n$ with the restriction $A = A_0$, $A_n = B, A_{i-1} \subset U_{bi-f\,\epsilon}(A_i)$ and $A_i \subset U_{bi-f\,\epsilon}(A_{i-1})$. If there exists a function $- f - \epsilon$ –chain amongst two sets A and B then $< A, B >$ is function – bi–$f - \epsilon$ –chainable and $< A, B >$ is function– bi – f –chainable when it is function– bi –$f - \epsilon$ –chainable for any $\epsilon > 0$.

Visibly each $U_{bi-f\,\epsilon}(x)$ is an open set.

The following definition and results proved by Vijeta (2020) related to bi–$f - \epsilon$ –chainablity are stated below. These terms and results established by Shrivastava *et al.* (2014) for topological space, have been extended to bitopological space.

Definition 4:

Considering a point $x \in X$ and set $A \subset X$, define

$[x]_{bi-f\epsilon} = \{y \in X | y \ is \ bi - f - \epsilon -$ chainable $to \ x\}\}$ and

$[A]_{bi-f\epsilon} = \{B \subset X | < A, B > is \ bi - f - \epsilon -$ chainable$\}$.

Note: $bi - f - \epsilon -$ chainability amongst any two elements or amongst any two subsets of X forms an equivalence relation on a bitopological space X and the disjoint sets signified by $[x]_{bi-f\epsilon}$ or $[A]_{bi-f\epsilon}$ are equivalence classes which partition the space X. The set $[x]_{bi-f\epsilon}$ is a clopen set.

Theorem 1:

Consider a subset A and x an element of bitopological space X.

If $< A, [x]_{bi-f\epsilon} >$ are bi–$f - \epsilon$ –chainable then A is a subset of $[x]_{bi-f\epsilon}$.

Also $[x]_{bi-f\epsilon}$ is the maximal set which is bi–$f - \epsilon$ –chainable to the set A and $\sup[A]_{bi-f\epsilon} = [x]_{bi-f\epsilon}$

Theorem 2:

If A is any subset of Xwhich is a bitopological space and x, y be any two elements of X and $< A, [x]_{bi-f\epsilon} >$and $< A, [y]_{bi-f\epsilon} >$are $bi - f - \epsilon$ –chainable sets then $[x]_{bi-f\epsilon} = [y]_{bi-f\epsilon}$.

Secondly $< [x]_{bi-f\epsilon}, [y]_{bi-f\epsilon} >$ are bi–$f - \epsilon$ –chainable iff $[x]_{bi-f\epsilon} = [y]_{bi-f\epsilon}$.

Self–$bi - f$ –Chainability

Definition 5:

Consider A to be any subset of bitopological space X. If for any positive number ϵ, there happen to exist a continuous function which should also be non-constant say $f:(X,\tau) \to [0,\infty)$ satisfying the condition that any pair of elements of A can be linked by a bi$-f-\epsilon$ –chain, then in this case, A is termed as self – bi $-f-\epsilon$ –chainable. A is termed to be self– bi$-f$ –chainable if A is self – bi$-f-\epsilon$ –chainable for any positive ϵ .

Observations:

- **i.** The set $[x]_{bi-f\epsilon}$ for any element x of X is all the time self– bi$-f-\epsilon$ –chainable.
- **ii.** Any bitopological space is bi$-f$ –chainable iff it is self– bi$-f$ –chainable.

Theorem 3:

A bitopological space is self– bi$-f$ –chainable if fall its subsets are self– bi$-f$ –chainable.

Theorem 4:

A bitopologicalspace is bi$-f$ –chainable iff all the subsets of the space are self– bi$-f$ –chainable.

Strongly Self–$bi - f$ –Chainability

Definition 6:

Let $A \subset X$. If for any positive number ϵ, there happens to be a continuous mapping $f:(X,\tau) \to [0,\infty)$ which is also non-constant with the condition that any pair of elements in A can be linked by an bi$-f-\epsilon$ –chain comprising of points of A only then A is termed as strongly self– bi$-f-\epsilon$ –chainable.

This set A is considered to be strongly self– bi$-f$ –chainable if it is strongly self– bi$-f-\epsilon$ –chainable for any positive number ϵ.

It can be seen clearly that for every positive $\epsilon$the notion of bi-$f-\epsilon$ –chainability and strongly self– bi$-f-\epsilon$ –chainability are one and the same.

Theorem 5:

Any set which is connected is strongly self– bi$-f$ –chainable if the f is non-constant function on the set.

Corollary:

Any connected space is always bi$-f$ $-$chainable.

Theorem 6:

Conside r A as a subset of a bitopological space (X, τ_1, τ_2). If A is self– bi$-f-\epsilon$ $-$chainable, then $\bar{A}$ is self– bi$-f-\epsilon$ $-$chainable.

Observation:

Self– bi$-f-\epsilon$ $-$chainablity of any set A is a resultant of self– bi$-f-\epsilon$ $-$chainablity of $\bar{A}$

Theorem 7:

For any two subsets A and B of a bitopological space X, following results hold true.

i. If A, B are mutually non-disjoint and are self$-bi-f$ $-$chainable then its union $A \cup B)$ is also self$-bi-f$ $-$chainable.

ii. If $\{A_\alpha\}_{\alpha\in\Lambda}$ are self$-bi-f$ $-$chainable subsets of X and $\bigcup_{\alpha\in\Lambda} A_\alpha \neq \emptyset$ then $\bigcup_{\alpha\in\Lambda} A_\alpha$ is self $-bi-f$ $-$chainable.

iii. If C is self$-bi-f$ $-$chainable and $\{C_z\}_{z\in\Lambda}$ be a structure of self$-bi-f-$ chainable sets satisfying the condition $(C \cap C_z) \neq \emptyset \quad \forall z \in \Lambda$ then $C \cup (\bigcup_{z\in\Lambda} C_z)$ is self $-bi-f$ $-$chainable.

iv. Consider A, B to be two mutually non-disjoint sets and are self$-bi-f$ $-$chainable in X then $< A, B >$)is bi$-f$ $-$chainable.

Strongly function$-\boldsymbol{\varepsilon}-\boldsymbol{f}-$ chainablity between two sets:

Definition 7:

For $\varepsilon > 0$, consider a non-constant continuous function$f: X \to [0\infty)$ and A, B be two subsets of X. Then $< A, B >$ is believed to be strongly – $bi-f-\varepsilon$ $-$chainable if and only if A and B are self $-f-\varepsilon$ $-$chainable and $< A, B >$ is $bi-f-\varepsilon$ $-$chainable.

$< A, B >$ is believed to be strongly– $bi-f$ $-$chainable if it is strongly– $bi-f-\varepsilon$ $-$chainable for every positive real numberε.

Next two theorems give the characterizations of strongly$-bi-f-\varepsilon$ $-$chainable sets.

Theorem 8:

$< A, B >$ is strongly$-bi-f-\varepsilon-$chainable, if and only if there is an $bi-f-\varepsilon-$chain between every point of A and every point of B, where A and B are two subsets of X.

Proof:

Let $< A, B >$ be strongly$-bi-f-\varepsilon-$chainable sets and let x be any element of A and y in B.

Then as$< A, B >$ is $bi-f-\varepsilon-$chainable, x is $bi-f-\varepsilon-$chainable to some point z of B.

By self $-bi-f-\varepsilon-$chainability of set of B , z is $bi-f-\varepsilon-$chainable to some point y.

Hence x and y are $bi-f-\varepsilon-$chainable.

On the other hand, let $bi-f-\varepsilon-$chain exist from every point of A to every point of B and vice-versa.

Then clearly $< A, B >$ is $bi-f-\varepsilon-$chainable.

Next if x and x' are any two points of A then both are $bi-f-\varepsilon-$chainable to every point of B and hence x and x' are $bi-f-\varepsilon-$chainable.

Equivalently, A is self$-bi-f-\varepsilon-$chainable.

Similarly, B is self$-bi-f-\varepsilon-$chainable.

Theorem 9:

X is strongly $bi-f-\varepsilon-$chainable if and only if, for A, B subsets of X, $A \cup B$ is self $bi-f-\varepsilon-$chainable.

Proof:

Consider $< A, B >$to be a strongly $bi-f-\varepsilon-$chainable setsand x and y be two points in $A \cup B$.

If x is in A and y in B then by previous theorem, there is an $bi-f-\varepsilon-$chain between these two points x and y. If x, y are in A or x, y are in B then since A and B are self $bi-f-\varepsilon-$chainable sets, a $bi-f-\varepsilon-$chain exists between x and y.

On the other hand, assume $A \cup B$ to be self $bi-f-\varepsilon-$chainable, x belongs to A and y belongs to B then $x, y \in A \cup B$ and hence a $bi-f-\varepsilon-$chain exists between x and y.

Equivalently, $< A, B >$is strongly $bi-f-\varepsilon-$chainable.

Theorem 10:

If $A \subset [x]_{f\varepsilon}$, for a subset A of X then $< A, [x]_{f\varepsilon} >$ is strongly $bi - f - \varepsilon$ −chainable.

Proof:

Assume $y \in A$ and $z \in [x]_{f\varepsilon}$. Then $y \in [x]_{f\varepsilon}$ and hence y and z are $bi - f - \varepsilon$ −chainable.

Equivalently, $< A, [x]_{f\varepsilon} >$ is strongly $bi - f - \varepsilon$ −chainable.

Observation:

The result given above confirms that converse of theorem 1 stands true.

Theorem 11:

Let A be self−$bi - f - \varepsilon$ −chainable subset of X. If $< [x]_{f\varepsilon}, A^c >$ is $bi - f - \varepsilon$ −chainable then $< A, [x]^c_{f\varepsilon} >$ is $bi - f - \varepsilon$ −chainable.

Proof:

Let $< A, [x]^c_{f\varepsilon} >$ is $bi - f - \varepsilon$ −chainable, then by previous theorem,

$A^c \subset [x]_{f\varepsilon}$ or $[x]^c_{f\varepsilon} \subset A$.

Since A is self−$bi - f - \varepsilon$ −chainable then $< A, [y]_{f\varepsilon} >$ is $bi - f - \varepsilon$ −chainable for any y in A.

Equivalently, A is a subset of $[y]_{f\varepsilon}$

Equivalently, $[x]^c_{f\varepsilon}$ is a subset of $[y]_{f\varepsilon}$.

Hence by theorem 3, $< [x]^c_{f\varepsilon}, [y]_{f\varepsilon} >$ is $bi - f - \varepsilon$ −chainable.

Equivalently, $< A, [x]^c_{f\varepsilon} >$ is $bi - f - \varepsilon$ −chainable.

Theorem 12:

A bitopologicalspace is $bi - f - \varepsilon$ −chainable for any positive number ε if and only if it is strongly− $bi - f - \varepsilon$ −chainable for any positive number ε.

Theorem 13:

Let A be self−$bi - f - \varepsilon$ −chainable, then for all $y \in A$,

$< A, [y]_{f\varepsilon} >$ is strongly− $bi - bi - f - \varepsilon$ −chainable and holds true conversely.

Conclusion:

In this article, the results related to strongly function chainability between sets, which were studied for topological space by Shrivastava, *et al.* (2014) have been extended and generalized for bitopological space.

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ANNIHILATOR SEMI-IDEALS IN A SEMILATTICE

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Abstract:

In this paper some properties of annihilator semi-ideals in a semilattice with 0 and annihilator preserving homomorphism are furnished.

Keywords: Distributive semilattice, 0- distributive semilattice, modular semilattice, semi-ideal.

AMS subject classification: 06A12, 06 B75, 06C99.

Introduction:

As a generalization of distributive semilattices introduced by Grätzer and Schmidt [3] and pseudo-complemented semilattices introduced by Frink [1], Varlet [7] has defined and studied 0-distributive semilattices.

Let $S = \langle S\,,\ \wedge \rangle$ be a semilattice with 0. For any non-empty subset A of S, the set of all annihilators of A is denoted by A^*. In a 0-distributive semilattice S, A^* is an ideal in the sense of Grätzer [2] for $\emptyset \neq A \subseteq S$ but it is not so in general. In any semilattice S with 0, A^* is a semi-ideal in the sense of Venkatnarsimhan [8] for any $\emptyset \neq A \subseteq S$. We call them annihilator semi-ideals. Some properties of these annihilator semi-ideals in S are studied in section 3.

For a congruence relation R defined on a semilattice S with 0, by $x \equiv y(R) \Leftrightarrow \{x\}^* = \{y\}^*\ (x, y \in S)$ it is proved that the quotient semilattice $\langle S\,/\,R\,,\ \wedge \rangle$ is isomorphic with the semilattices $\langle S^{**},\ \cap \rangle$ and $\langle \mathcal{K},\ \cap \rangle$ where $S^{**} = \{\{x\}^{**} \mid x \in S\}$ and $\mathcal{K} = \{\{M\}_x \mid x \in S\}$, here $\{M\}_x = \{M \mid x \notin M,\ M \text{ is a minimal prime semi-ideal in } S\}$. Further it is proved that the semilattice $\langle S^{**},\ \cap \rangle$ is a Boolean algebra if and only if for any non-empty subset A of S, there exists $a' \in S$ such that $A^{**} = \{a'\}^*$.

In section 4 the concept of an annihilator preserving homomorphism is introduced in a natural way. Some properties of annihilator homomorphism are proved. Mainly we prove that given a semi-ideal I of a semilattice S with 0 there exists a semilattice S' with $0'$ and a homomorphism $\theta: S \to S'$ such that $\theta(0) = 0'$ and $Ker\theta = I^*$.

Preliminaries:

This section is devoted to a summary of known concepts and results which will be used in the subsequent sections. For basic concepts in lattice theory, the reader is referred to [2]. Throughout this paper we shall be concerned with $\wedge -$ semilattice $\langle S, \wedge \rangle$ which will be simply denoted by S only. For notions and notations in semilattices we follow [4] and [8].

The partial ordering $\leq$ in S is defined by $a \leq b$ if and only if $a \wedge b = a$. The least (greatest) element is denoted by 0 (1), when it exists. A semilattice S is said to be bounded if both 0 and 1 exist in S. Now onwards S will denote a meet semilattice with 0 unless otherwise stated. A non-empty subset A of S is called a semi-ideal if $a \in A, b \leq a\ (b \in S)$ implies $b \in A$. A proper semi-ideal A of S is said to be prime if $a \wedge b \in A$ implies $a \in A$ or $b \in A$. A prime semi-ideal is called minimal prime semi-ideal if it does not contain any other prime semi-ideal.

An ideal I of S is a semi-ideal I of S such that for any $x, y \in I$ there exists $t \in I$ such that $t \geq x$ and $t \geq y$. An ideal $I \neq S$ is called a proper ideal. A proper ideal I is said to be prime ideal of S if $x \wedge y \in I$ imply $x \in I$ or $y \in I$ $(x, y \in S)$. A proper ideal M of S is said to be maximal if it is not contained in any other proper ideal. For any subset H of S the smallest ideal containing H is called the ideal generated by H and is denoted by $(H]$. The principal ideal generated by $a \in S$ is the set $\{x \in S : x \leq a\}$ and is denoted by $(a]$.

A filter of S is a non-empty subset F of S such that for any $x, y \in F$ we have $x \wedge y \in F$ and $x \in F, y \geq x\ (y \in S)$ imply $y \in F$. A filter F of S is said to be proper if $F \neq S$. A proper filter F of S is said to maximal if it is not contained in any other proper filter. For any non-empty subset H of S the smallest filter containing H is called the filter generated by H and it is denoted by $[H)$. The principal filter generated by $a \in S$ is the set $\{x \in S : x \geq a\}$ and it is denoted by $[a)$. For any non-empty subset A of S, the set $A^* = \{x \in S : x \wedge a = 0 \text{ for all } a \in A\}$ is a called annihilator of A.

Definition 1: S is said to be pseudo-complemented if for every $a \in S$ there exists $b \in S$ such that $\{a\}^* = (b]$. This b is called the pseudo-complement of a and is denoted by a^*.

Definition 2: S is said to be a * - semilattice if for any $x \in S$ there exists $x' \in S$ such that $\{x\}^{**} = \{x'\}^*$.

Definition 3: S is called a modular semilattice if $x, y, z \in S$ and $x \geq y \wedge z$ imply the existence of $t, r \in S$ such that $x \wedge t = x \wedge r = t \wedge r$.

Definition 4: S is said to be weakly distributive semilattice if $\{x\}^*$ is an ideal for any $x \in S$.

Definition 5: S is called 0-distributive semilattice if A^* is an ideal for any non-empty subset A of S.

Definition 6: S is said to be disjunctive (weakly complemented) if for $a, b \in S$ and $a \gneqq b$ imply that there exists $c \in S$ such that $a \wedge c = 0$ but $b \wedge c \neq 0$.

Definition 7: S is said to be dense if $(a]^* = (0]$ for all $0 \neq a \in S$.

We collect below some known results used in sequel.

Result 2.1: In S, a proper filter F is maximal if and only if for $x \notin F$, there exists $y \in F$ such that $x \wedge y = 0$.

Result 2.2: A non-empty proper subset F of S is a maximal filter if and only if $S \backslash F$ is a minimal prime semi-ideal of S.

Result 2.3: Let $\mathfrak{M}$ denote the set of minimal prime semi-ideals of S. Define $\{M\}_x = \{M \in \mathfrak{M} \mid x \notin M\}$. Then $\{M\}_x \cap \{M\}_y = \{M\}_{x \wedge y}$ for any $x, y \in S$.

Result 2.4: A prime semi-ideal A of S is minimal prime if and only if A contains precisely one of $\{x\}, \{x^*\}$ for every x in S.

Annihilators in semilattices:

The aim of this section is to prove various properties of the annihilators in S. Obviously for any non-empty A of S, A^* is semi-ideal of S. This semi-ideal A^* is called annihilator semi-ideal. It has following properties which can be verified directly by the definition.

Lemma 3.1: Following properties hold in S for annihilator semi-ideals in S. Let A and B denote non-empty subsets of S.

(1) $A^* = \bigcap\{\{a\}^* \mid a \in A\}$.

(2) If $A \subseteq B$, then $B^* \subseteq A^*$ and $A^{**} \subseteq B^{**}$.

(3) $A \subseteq A^{**}$.

(4) $A^* = A^{***}$.

(5) $A^* \subseteq B^* \iff B^{**} \subseteq A^{**}$.

(6) $A^* \cap A^{**} = \{0\}$.

(7) If I and J are semi-ideals then $I \cap J = (0] \iff I \subseteq J^*$.

Following is an extension of the result of Speed (see [6], Theorem 2) to semi-ideals in a semilattice.

Theorem 3.2: $(I \cap J)^{**} = I^{**} \cap J^{**}$ for any two semi-ideals I and J of S.

Proof: $(I \cap J)^{**} \subseteq I^{**} \cap J^{**}$ follows by Lemma 3.1 (2). Let $x \in I^{**} \cap J^{**}$. For any $a \in I$ and $b \in J$, $a \wedge b \in I \cap J$ since I and J are semi-ideals. Hence $a \wedge b \wedge t = 0$ for each $t \in (I \cap J)^*$ will imply $a \wedge t \in \bigcap\{\{b\}^* \mid b \in J\} = J^*$ (see Lemma 3.1,(1)). As $x \in J^{**}$, we get $a \wedge t \wedge x = 0$. Thus $t \wedge x \in \bigcap\{\{a\}^* \mid a \in I\} = I^*$ (see Lemma 3.1, (1)). As $t \wedge x \leq x$ and $x \in I^{**}$ we get $t \wedge x \in I^{**}$, I^{**} being a semi-ideal of S. Thus $t \wedge x \in I^* \cap I^{**} = \{0\}$ implies $t \wedge x = 0$ i.e. $x \in \{t\}^*$. As this is true for any $t \in (I \cap J)^*$, we get $x \in \bigcap\{\{t\}^* \mid t \in (I \cap J)^*\} = (I \cap J)^{**}$ by Lemma 3.1 (1). Hence $I^{**} \cap J^{**} \subseteq (I \cap J)^{**}$. Combining both the inclusions, the result follows.

More generally we have

Corollary 3.3: If $\{I_\alpha \mid \alpha \in \Delta\}$ is a family of semi-ideals of S (Δ is any indexing set), then

$$\left[\bigcap_{\alpha \in \Delta} I_\alpha\right]^{**} = \bigcap_{\alpha \in \Delta} {I_\alpha}^{**}.$$

For any $a \in S, (a]^* = \{a\}^*$. As a special case of Theorem 3.2 we have

Corollary 3.4: $\{a \wedge b\}^{**} = \{a\}^{**} \cap \{b\}^{**}$ for all $a, b \in S$.

Define a relation R on S as follows:

$$x \equiv y(R) \Leftrightarrow \{x\}^* = \{y\}^* (x, y \in S).$$

By Corollary 3.4, R is a congruence relation on S. Hence the $\wedge$ - semilattice $\langle S/R\ ,\ \wedge\rangle$ is defined where $S/R = \{[x]^R \mid x \in S\}$ and $\wedge$ on S/R is defined by $[x]^R \wedge [y]^R = [x \wedge y]^R$, for $x, y \in S$. Also by Corollary 3.4, we get $\langle S^{**}\ ,\ \cap\rangle$ is a semilattice where $S^{**} = \{\{x\}^{**} \mid x \in S\}$. Interestingly these two $\wedge$ - semilattices S/R and S^{**} are isomorphic. This we prove in the following theorem.

Theorem 3.5: For a semilattice S, $\langle S/R\ ,\ \wedge\rangle$ is isomorphic with $\langle S^{**}\ ,\ \cap\rangle$.

Proof: Define $\theta: S/R \longrightarrow S^{**}$ by $\theta([x]^R) = \{x\}^{**}$ for each $x \in S$. Obviously θ is well defined, onto map. For $x, y \in S$, let $\theta([x]^R) = \theta([y]^R)$. But then $\{x\}^{**} = \{y\}^{**} \Longrightarrow \{x\}^* = \{y\}^*$ (by Lemma 3.1 (5)). Thus $x \equiv y(R)$. Hence $[x]^R = [y]^R$. This shows that θ is one-one.

Again for $x, y \in S$, $\theta([x]^R \wedge [y]^R) = \theta([x \wedge y]^R) = \{x \wedge y\}^{**} = \{x\}^{**} \cap \{y\}^{**}$ (see the Corollary 3.4). Hence $\theta([x]^R \wedge [y]^R) = \theta([x]^R) \cap \theta([y]^R)$. This shows that θ is a homomorphism. θ being an isomorphism we get $S/R \cong S^{**}$. □

It is well known that there is a close connection between annihilator semi-ideals and minimal prime semi-ideals of S. To support this statement we quote some results from [8].

Let $\mathfrak{M}$ denote the set of all minimal prime semi-ideals of S.

1) $A \in S_\mu \Rightarrow A^* = \cap \{M \in \mathfrak{M} \mid A \nsubseteq M\}$.

2) $A \in S_\mu$ and $A^* \neq \{0\} \Rightarrow A \subseteq M$ for some $M \in \mathfrak{M}$.

3) If $a \in M$ $(M \in \mathfrak{M})$, then $\{a\}^* \neq \{0\}$.

4) $A \in S_\mu$ and $A = A^{**} \Rightarrow A = \bigcap\{M \in \mathfrak{M} \mid A \subseteq M\}$.

Further we have

Theorem 3.6: Let F be a filter of S such that $\{x\}^* \cap F = \emptyset$, for some $x \in S$. Then there exists $M \in \mathfrak{M}$ containing $\{x\}^*$ and disjoint with F.

Proof: The existence of a filter Q maximal with respect to the property of containing F and disjoint with $\{x\}^*$ follows by Zorn's lemma. If $x \notin Q$, then $[Q \vee [x)] \cap \{x\}^* \neq \emptyset$ by the choice of Q. Hence there exists $t \in \{x\}^*$ such that $t \geq q \wedge x$ for some $q \in Q$. But then $t \wedge x = 0$ will imply $q \in Q \cap \{x\}^* = \emptyset$; which is absurd. Hence $x \in Q$. Again for any $y \notin Q$ we have $[Q \vee [y)] \cap \{x\}^* \neq \emptyset$, by the choice of Q. Hence there exists $s \in \{x\}^*$ such that $s \geq r \wedge y$ for some $r \in Q$. But then $s \wedge x = 0$ implies $r \wedge y \wedge x = 0$. As $x \in Q$ and $r \in Q$, $r \wedge x \in Q$. Thus for $y \notin Q$, there exists $r \wedge x \in Q$ such that $y \wedge (r \wedge x) = 0$. Hence Q is a maximal filter in S (see Result 2.1). Define $M = S \backslash Q$. Then M is minimal prime semi-ideal containing $\{x\}^*$ and disjoint with F (see Result2.2).

Define $\mathcal{K} = \{\{M\}_x \mid x \in S\}$ where $\{M\}_x = \{M \in \mathfrak{M} \mid x \notin M\}$. Then $\langle \mathcal{K}, \cap \rangle$ is a semilattice as $\{M\}_x \cap \{M\}_y = \{M\}_{x \wedge y}$ (see Result 2.3) for any $x, y \in S$.

Now we prove that for a semilattice S the three semilattices $\langle {}^S/_R\ , \wedge \rangle$, $\langle \mathcal{K}, \cap \rangle$ and $\langle S^{**}, \cap \rangle$ are isomorphic. For this we first prove

Theorem 3.7: For S, the semilattices $\langle {}^S/_R\ , \wedge \rangle$ and $\langle \mathcal{K}, \cap \rangle$ are isomorphic.

Proof: Define $\psi : {}^S/_R \longrightarrow \mathcal{K}$ by $\psi([x]^R) = \{M\}_x$.

Claim 1: $\{M\}_x = \{M\}_y \iff \{x\}^* = \{y\}^*$ for $x, y \in S$.

Let $x, y \in S$ such that $\{x\}^* \neq \{y\}^*$. Without loss of generality assume that there exists $z \in \{x\}^*$ such that $z \notin \{y\}^*$. As $[z) \cap \{y\}^* = \emptyset$, by Theorem 3.6, there exists a minimal prime semi-ideal M containing $\{y\}^*$ and not containing z. As $z \notin M$ and $z \in \{x\}^*$ we get $\{x\}^* \nsubseteq M$. Applying Result 2.4 we get $\{y\}^* \subseteq M \Rightarrow y \notin M$ and $\{x\}^* \nsubseteq M \Rightarrow x \in M$. Thus $M \in \{M\}_y$ and $M \notin \{M\}_x$ imply $\{M\}_x \neq \{M\}_y$. Thus $\{x\}^* \neq \{y\}^* \Rightarrow \{M\}_x \neq \{M\}_y$. Let us assume that for some $x, y \in S$, $\{M\}_x \neq \{M\}_y$. Without loss of generality assume that there exists $M \in \mathfrak{M}$ such that

$M \in \{M\}_x$ but $M \notin \{M\}_y$. Then $x \notin M \Longrightarrow \{x\}^* \subseteq M$ and $y \in M \Longrightarrow \{y\}^* \nsubseteq M$ (see Result 2.4). Thus $\{M\}_x \neq \{M\}_y \Longrightarrow \{x\}^* \neq \{y\}^*$. Hence $\{M\}_x = \{M\}_y \iff \{x\}^* = \{y\}^*$ for all $x, y \in S$.

Claim 2: ψ is well defined.

Let $[x]^R = [y]^R$ for some $x, y \in S$. Then $x \equiv y(R) \Longrightarrow \{x\}^* = \{y\}^* \Longrightarrow \{M\}_x = \{M\}_y$ (by Claim 1). Therefore $[x]^R = [y]^R \Longrightarrow \psi([x]^R) = \psi([y]^R)$. Hence ψ is a well defined map.

Claim 3: ψ is one-one and onto map .

Obviously ψ is an onto map. To prove that ψ is one-one let $\psi([x]^R) = \psi([y]^R)$ for $x, y \in S$. Then $\{M\}_x = \{M\}_y \Longrightarrow \{x\}^* = \{y\}^*$(by Claim 1). By definition of R, $\{x\}^* = \{y\}^* \Longrightarrow x \equiv y(R) \Longrightarrow [x]^R = [y]^R$. Thus $\psi([x]^R) = \psi([y]^R) \Longrightarrow [x]^R = [y]^R$ for $x, y \in S$. This shows that ψ is one-one.

Claim 4: ψ is a homomorphism.

Let $x, y \in S$. $\psi([x]^R \wedge [y]^R) = \psi([x \wedge y]^R) = \{M\}_{x \wedge y} = \{M\}_x \cap \{M\}_y$ (by Result 2.3). Thus $\psi([x]^R \wedge [y]^R) = \psi([x]^R) \cap \psi([y]^R)$ for all $x, y \in S$. This shows that ψ is a homomorphism. Thus we get ψ an isomorphism and hence the semilattices ${}^{S}/_{R}$ and $\mathcal{K}$ are isomorphic. □

As ${}^{S}/_{R} \cong S^{**}$ and ${}^{S}/_{R} \cong \mathcal{K}$, we get

Corollary 3.8: For a semilattice S, the three semilattices $\langle {}^{S}/_{R}\ ,\ \wedge \rangle$, $\langle \mathcal{K},\ \cap \rangle$ and $\langle S^{**},\ \cap \rangle$ are isomorphic.

Let F be a filter in S. Then a relation $\theta(F)$ defined on S by

$$x \equiv y\big(\theta(F)\big) \Leftrightarrow x \wedge f = y \wedge f \text{ for some } f \in F$$

is a congruence relation on S.

Definition: An ideal I of S is an α – ideal if $\{x\}^{**} \subseteq I$ for each $x \in S$.

In a * - semilattice for an α – ideal we have

Theorem 3.9: Let S be a * - semilattice. Then for each α - ideal I of S,there exists a filter F in S such that $I = Ker\big(\theta(F)\big)$.

Proof: Let I be an α – ideal in a * - semilattice S.

Define $F = \{x \in S : \{z\}^* \subseteq \{x\}^{**}\text{for some } z \in I\}$.

Claim 1: F is a filter in S.

$0 \in J$ and $\{0\}^* = S \subseteq \{1\}^{**}$. Therefore $1 \in F$. Let $x \in F$ and $x \leq y$ for some y in S. Then $x \leq y \Longrightarrow \{x\}^{**} \subseteq \{y\}^{**}$(see Lemma 3.1 (2)). As $x \in F$ there exists $z \in I$ such that $\{z\}^* \subseteq \{x\}^{**}$. Hence $\{z\}^* \subseteq \{y\}^{**} \Longrightarrow y \in F$. Again let$x_1$, $x_2 \in F$. Then there exist f_1 , $f_2 \in I$ such that $\{f_1\}^* \subseteq \{x_1\}^{**}$ and $\{f_2\}^* \subseteq \{x_2\}^{**}$. As I is ideal in S, f_1 , $f_2 \in I$ implies there exists $t \in I$ such that $t \geq f_1$ and

$t \geq f_2$. But then $\{t\}^* \subseteq \{f_1\}^* \cap \{f_2\}^* \subseteq \{x_1\}^{**} \cap \{x_2\}^{**} = \{x_1 \wedge x_2\}^{**}$ (see Corollary 3.4). This shows that $x_1 \wedge x_2 \in F$. Hence F is a filter in S.

Claim 2: $I = Ker(\theta(F))$.

Let $x \in I$. As S is a * - semilattice, there exists $x' \in S$ such that $\{x\}^{**} = \{x'\}^*$. Hence $x \wedge x' = 0$. Further $\{x\}^* = \{x'\}^{**} \Rightarrow x' \in F$. Thus $x \wedge x' = 0 = 0 \wedge x'$ and $x' \in F$ $\Rightarrow x \equiv 0(\theta(F))$. Hence $x \in Ker(\theta(F))$. Therefore $I \subseteq Ker(\theta(F))$. Now let $x \in Ker(\theta(F))$. Then $x \equiv 0(\theta(F)) \Rightarrow x \wedge f = 0$ for some $f \in F$. Hence there exists $z \in I$ such that $\{z\}^* \subseteq \{f\}^{**}$. Thus $x \in \{f\}^* \subseteq (z]^{**}$. Since I is an α–ideal, $z \in I \Rightarrow (z]^{**} \subseteq I$.

Thus $x \in Ker(\theta(F)) \Rightarrow x \in I$. This shows that $Ker(\theta(F)) \subseteq I$. Combining both the inclusions we get $I = Ker(\theta(F))$. □

Converse of Theorem 3.9 is true if S is a $0-$ distributive semilattice.

Theorem 3.10: Let S be a 0-distibutive semilattice. If for each $\alpha -$ ideal I there exists a filter F in S such that $I = Ker(\theta(F))$, then S is a * - semilattice.

Proof: Select $x \in S$. As S is a 0-distibutivesemilattice, $\{x\}^{**}$ is an ideal in S. But $\{x\}^{**} = (x]^{**}$ being an $\alpha -$ ideal in S, by assumption, there exists a filter F of S such that $\{x\}^{**} = Ker(\theta(F))$.

Now $t \in Ker(\theta(F)) \Leftrightarrow t \equiv 0(\theta(F))$.

$\Leftrightarrow t \wedge f = 0 \wedge f$ for some $f \in F$.

$\Leftrightarrow t \wedge f = 0$ for some $f \in F$.

$\Leftrightarrow t \in \{f\}^*$ for some $f \in F$.

Hence $Ker(\theta(F)) = \bigcup\{\{f\}^* \mid f \in F\}$. But then $\{x\}^{**} = \bigcup\{\{f\}^* \mid f \in F\}$.

As $x \in \{x\}^{**} \Rightarrow x \in \{y\}^*$ for some $y \in F$.Hence $\{x\}^{**} \subseteq \{y\}^{***} = \{y\}^*$.

But $y \in F \Rightarrow \{y\}^* \subseteq \{x\}^{**}$. Thus $\{x\}^{**} = \{y\}^*$. Hence S is a * - semilattice. □

A necessary and sufficient condition for S to be a * - semilattice is given in the following theorem.

Theorem 3.11: S is a * - semilattice if and only if the semilattice $\langle S^{**}, \cap \rangle$ is a Boolean algebra.

Proof: To Prove that S is a * - semilattice. Let $a \in S$. Then $\{a\}^{**} \in S^{**}$. As S^{**} is complemented, $[\{a\}^{**}]' \in S^{**}$. Let $[\{a\}^{**}]' = \{a'\}^{**}$ for some $a' \in S$. Then $\{a\}^{**} \cap \{a'\}^{**} = \{0\}^{**} \Rightarrow \{a \wedge a'\}^{**} = \{0\}^{**} = \{0\}$ (by Corollary 3.4)

$\Rightarrow a \wedge a' = 0 \Rightarrow a' \in \{a\}^* \Rightarrow \{a'\}^* \supseteq \{a\}^{**}$ _________ (1)

Now let $b \in \{a'\}^*$. Then

$\{b\} \subseteq \{a'\}^* \Rightarrow \{b\}^{**} \subseteq \{a'\}^{***} = \{a'\}^*$ (Lemma 3.1 (2) and 3.1 (4))

$\Rightarrow \{b\}^{**} \cap \{a'\}^{**} = \{0\}^{**}$ (by Lemma3.1 (6))

$\Rightarrow \{b\}^{**} \cap \{a\}^{**} = \{b\}^{**}$ (since $\{a'\}^{**} = [\{a\}^{**}]'$)

$\Rightarrow \{b \wedge a\}^{**} = \{b\}^{**}$ (by Corollary 3.4)

$\Rightarrow b \leq a \Rightarrow b \in \{a\}^{**}$

Thus $b \in \{a'\}^* \Rightarrow b \in \{a\}^{**}$. Hence $\{a'\}^* \subseteq \{a\}^{**}$ ___________ (2)

From (1) and (2) we get $\{a'\}^* = \{a\}^{**}$. This shows that for any $a \in S$ there exists $a' \in S$ such that $\{a\}^{**} = \{a'\}^*$. Hence S is a * - semilattice.

Conversely, suppose S is a * - semilattice. To prove that S^{**} is a Boolean algebra.

Let $\{a\}^{**} \in S^{**}$. Then $a \in S \Rightarrow$ there exits $a' \in S$ such that $\{a\}^{**} = \{a'\}^*$.

Define $[\{a\}^{**}]' = \{a'\}^{**}$. As S^{**} is a semilattice we will only verify that

$\{b\}^{**} \cap [\{a\}^{**}]' = \{0\}^{**} \Leftrightarrow \{b\}^{**} \cap \{a\}^{**} = \{b\}^{**}$ for any $b^{**} \in S^{**}$.

I] $\{b\}^{**} \cap [\{a\}^{**}]' = \{0\}^{**}$

$\Rightarrow \{b\}^{**} \cap \{a'\}^{**} = \{0\}^{**}$ (Since$[\{a\}^{**}]' = \{a'\}^{**}$)

$\Rightarrow \{b \wedge a'\}^{**} = \{0\}^{**} = \{0\}$ (See Corollary 3.4)

$\Rightarrow b \wedge a' = 0$

$\Rightarrow a' \in \{b\}^*$

$\Rightarrow \{a'\}^* \supseteq \{b\}^{**}$(By Lemma 3.1 (2))

$\Rightarrow \{a\}^{**} \supseteq \{b\}^{**}$ (Since$\{a\}^{**} = \{a'\}^*$)

$\Rightarrow \{a\}^{**} \cap \{b\}^{**} = \{b\}^{**}$

II] Let$\{a\}^{**} \cap \{b\}^{**} = \{a\}^{**}$. Then $\{a \wedge b\}^{**} = \{a\}^{**}$ ______________ (3)

Hence $\{b\}^{**} \cap [\{a\}^{**}]' = \{b\}^{**} \cap \{a'\}^{**}$

$= \{a \wedge b\}^{**} \cap \{a'\}^{**}$… (by (3))

$= \{a \wedge b \wedge a'\}^{**}$.

Now $a \wedge a' \in \{a\}^{**}$ (since $a \in \{a\}^{**}$ and$\{a\}^{**}$ is a semi-ideal).

Similarly $a \wedge a' \in \{a'\}^{**}$. Hence $a \wedge a' \in \{a\}^{**} \cap \{a'\}^{**} = \{a\}^{**} \cap [\{a\}^{**}]' = \{0\}$

$\Rightarrow a \wedge a' \in \{0\}$ and hence $a \wedge a' = 0$.

But then $b \wedge a \wedge a' = 0$ shows that $\{b\}^{**} \cap [\{a\}^{**}]' = \{0\} = \{0\}^{**}$.

From I] and II] we get

$$\{b\}^{**} \cap [\{a\}^{**}]' = \{0\}^{**} \Leftrightarrow \{a\}^{**} \cap \{b\}^{**} = \{a\}^{**}$$

Therefore S^{**} is a Boolean algebra (see Frink [1]). □

Using Theorem 3.11 we characterize complete Boolean algebra S^{**}as follows.

Theorem 3.12: For S, the semilattice $\langle S^{**}, \cap \rangle$ is a complete Boolean algebra if and only if for any non-empty subset A of S, there exists $a' \in S$ such that $A^{**} = \{a'\}^*$.

Proof: Only if part: Let S^{**} be a complete Boolean algebra and $A \neq \emptyset$ be a subset of S. Then $A^* = \bigcap\{\{a\}^* \mid a \in A\}$ (see Lemma 3.1 (1)). As S^{**} is Boolean algebra, by Theorem 3.11, S is a *-semilattice. Hence for every $a \in A$ there exists $a' \in S$ such that $\{a\}^{**} = \{a'\}^*$ or equivalently $\{a\}^* = \{a'\}^{**}$. Therefore $A^* = \bigcap\{\{a'\}^{**} \mid \{a\}^* = \{a'\}^{**}, a \in A\}$. As S^{**} is complete, $\bigcap\{\{a'\}^{**} \mid \{a\}^* = \{a'\}^{**}, a \in A\} \in S^{**}$. Let $\bigcap\{\{a'\}^{**} \mid \{a\}^* = \{a'\}^{**}, a \in A\} = \{b\}^{**}$ for some $b \in S$. This shows that $A^* = \{b\}^{**}$i.e. $A^{**} = \{b\}^*$ and the result follows.

If part: Let the condition be satisfied by S. To prove that $\langle S^{**}, \cap \rangle$ is a complete Boolean algebra. By the given condition, S is a * - semilattice. Hence by Theorem 3.11, S^{**} is a Boolean algebra. To prove that S^{**} is a complete lattice. For this consider any subset $\{\{a_\alpha\}^{**} \mid \alpha \in \Delta\}$ (Δ is any indexing set) of S^{**}. For every $\alpha \in \Delta$, $\{a_\alpha\}^{**} = \{a'_\alpha\}^*$ for some $a'_\alpha \in S$ (by assumption). Hence we get $\bigcap\{\{a_\alpha\}^{**} \mid \alpha \in \Delta\} = \bigcap\{\{a'_\alpha\}^* \mid \alpha \in \Delta\}$. Let $A = \{a'_\alpha \mid \alpha \in \Delta\}$. Then $A^* = \bigcap\{\{a'_\alpha\}^* \mid \alpha \in \Delta\}$ (by Lemma 3.1 (1)). But by assumption there exists $b \in S$ such that $A^* = \{b\}^{**}$. Thus $A^* \in S^{**} \Longrightarrow \bigcap\{\{a_\alpha\}^{**} \mid \alpha \in \Delta\} \in S^{**}$. Therefore S^{**} is a complete lattice (see [2] Lemma 14). □

Annihilator preserving homomorphism:

In this section we prove various properties of an annihilator preservinghomomorphism.

Throughout this article S and S' denote semilattices with zero elements 0 and $0'$ respectively. For a homomorphism $f: S \to S'$ following results can be verified easily.

(1) If f is onto, then for any semi-ideal I of S, $f(I)$ is an semi-ideal of S'.

(2) For any semi-ideal J of S', $f^{-1}(J)$ is a semi-ideal of S containing $Kerf$, where $Kerf = \{x \in S : f(x) = 0'\}$.

(3) For any non-empty subset A of S we have $f(A^*) \subseteq [f(A)]^*$.

But $[f(A)]^* \subseteq f(A^*)$ is not true in general. For this consider the semilattice S whose diagrammatic representation is as shown in the Figure 1.

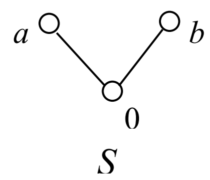

Fig. 1

Define the mapping $f: S \to S$ by $f(x) = 0$ for all $x \in S$.

Then f is a homomorphism on S.

Take $A = \{0, a\}$, then $A^* = \{0, b\}$ and $f(A) = \{0\}$.

Hence $f(A^*) = \{0\}$ and $[f(A)]^* = S$.

This shows that $[f(A)]^* \nsubseteq f(A^*)$.

This leads us to define annihilator preservinghomomorphism.

Definition 4.1: A homomorphism $f: S \to S'$ is called annihilator preserving if $f(A^*) = [f(A)]^*$ for any $(0] \subset A \subset S$.

Example4.2: Let S and S' be semilattices whose diagrammatic representations are as shown in Figure 2.

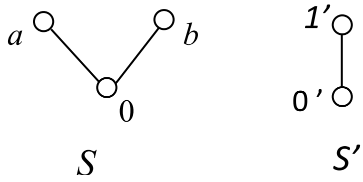

Fig. 2

Define $f: S \to S'$ by

$f(0) = 0' = f(b),\ f(a) = 1'$.

Then f is an annihilator preserving homomorphism.

Every semilattice homomorphism $f: S \to S'$ need not be an annihilator preserving but if both semilattices S and S' are dense, then we have

Theorem 4.3: Let S and S' be two dense semilattices. Then every homomorphism from S into S' is annihilator preserving.

Proof: Let $f: S \to S'$ be a homomorphism. Let $(0] \subset A \subset S$.

$$f(A^*) = f\left(\bigcap_{a \in A} (a]^*\right) \left(\text{by Lemma 3.1, (1)}\right).$$

$$= f((0]^* \cap (0]) \ (\text{since} (a]^* = (0] \text{ for all } 0 \neq a \in S).$$

$$= f(S \cap (0]) \ (\text{since} (0]^* = S).$$

$$= f\big((0]\big)$$

$$= (0'] \text{ ________________ } (1).$$

Further $[f(A)]^* = \cap_{a \in A} (f(a)]^* \left(\text{by Lemma 3.1 (1)}\right)$

$$= (0']^* \cap (0'] \ (\text{since } (f(a)]^* = (0'] \text{ for all} 0' \neq f(a) \in S').$$

$$= S' \cap 0' \ (\text{since} (0']^* = S).$$

$$= (0'] \text{ ________________ } (2).$$

From (1) and (2) we get $f(A^*) = [f(A)]^*$, for every $(0] \subset A \subset S$. This shows that f is annihilator preserving. □

Unlike in rings, if $f: S \to S'$ is a homomorphism such that $Kerf = \{0\}$, then f need not be one-one. It can be seen by the following example.

Example 4.4: Consider the semilattices S and S' as shown in Figure 3.

Let $f: S \to S'$ be defined as,

$f(0) = 0', f(a) = f(1) = 1'$.

Then f is a homomorphism. Also $Kerf = \{0\}$.

But f is not one-one.

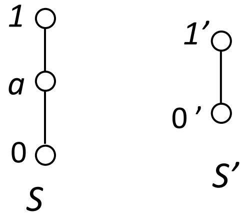

Fig. 3

A sufficient condition for a homomorphism to be an annihilator preserving map is given in the following theorem.

Theorem 4.5: Let $f: S \rightarrow S'$ be an epimorphism such that $Kerf = \{0\}$. Then both f and f^{-1} are annihilator preserving and for any $(0] \subset A \subset S$ and $(0] \subset B \subset S$, $A^* = B^*$ if and only if $[f(A)]^* = [f(B)]^*$.

Proof:

(1) Let $(0] \subset A \subset S$. Always we have $f(A^*) \subseteq [f(A)]^*$. As f is onto $f(S) = S'$. Let $f(t) \in [f(A)]^*$ for some $t \in S$. Then $f(t) \wedge f(a) = 0'$ for all $a \in A \Rightarrow f(t \wedge a) = 0'$ $\Rightarrow t \wedge a \in Kerf = \{0\} \Rightarrow t \wedge a = 0 \Rightarrow t \in \{a\}^*$ for all $a \in A$. Thus $t \in \bigcap\{\{a\}^* \mid a \in A\} \Rightarrow t \in A^*$ (see Lemma 3.1, (1)). Hence $f(t) \in f(A^*)$ which shows that $[f(A)]^* \subseteq f(A^*)$. Combining both the inclusions we get $f(A^*) = [f(A)]^*$. This proves that f is anannihilator preserving homomorphism.

(2) Let $(0] \subset B \subset S'$. Let $x \in [f^{-1}(B)]^*$ for some $x \in S$. Then $x \wedge b = 0$ for all $b \in f^{-1}(B) \Rightarrow x \wedge b = 0$ for all $b \in f^{-1}(B)$. As f is a homomorphism, $f(x) \wedge f(b) = 0'$ for all $f(b) \in B$. Therefore $f(x) \in \{ f(b)\}^*$ for all $f(b) \in B$.

Thus $f(x) \in \bigcap\{\{f(b)\}^* \mid f(b) \in B\} \Rightarrow f(x) \in B^*$ (see lemma 3.1, (1)) i.e. $x \in f^{-1}(B^*)$. Hence $[f^{-1}(B)]^* \subseteq f^{-1}(B^*)$.

Conversely, let $x \in f^{-1}(B^*)$ and $b \in f^{-1}(B)$ for some $x, b \in S$. Then $f(x) \in B^*$ and $f(b) \in B \Rightarrow f(x) \wedge f(b) = 0' \Rightarrow f(x \wedge b) = 0' \Rightarrow x \wedge b \in Kerf = \{0\} \Rightarrow x \wedge b = 0$ for any $b \in f^{-1}(B) \Rightarrow x \in [f^{-1}(B)]^*$. Hence $f^{-1}(B^*) \subseteq [f^{-1}(B)]^*$. Combining both the inclusions, the result follows.

(3) Assume that $A^* = B^*$. Then by (1) f being annihilator preserving homomorphism, we get $f(A^*) = f(B^*)$. This will imply $[f(A)]^* = [f(B)]^*$.

Conversely, assume that $[f(A)]^* = [f(B)]^*$. Let $x \in A^*$ for some $x \in S$. Then $x \wedge a = 0$ for all $a \in A \Rightarrow f(x) \wedge f(a) = 0'$ for all $f(a) \in f(A)$. But then $f(x) \in [f(A)]^* = [f(B)]^*$ will imply $f(x) \wedge f(b) = 0'$ for all $f(b) \in f(B)$. Thus $f(x \wedge b) = 0' \Rightarrow x \wedge b \in Kerf = \{0\}$. This shows that $A^* \subseteq B^*$. Similarly we can prove $B^* \subseteq A^*$. Hence $A^* = B^*$. □

Remarks 4.6: Let $f: S \rightarrow S'$ be a homomorphism. Then by the definition of a normal semi-ideal of S, we have the following.

(1) If f is annihilator preserving epimorphism then $f(I)$ is a normal semi-ideal of S' for every a normal semi-ideal I of S.

(2) If f^{-1} preserves annihilators, then $f^{-1}(J)$ is a normal semi-ideal of S for every normal semi-ideal J of S'.

(3) If f preserves annihilators, then $Ker f$ is a normal semi-ideal of S.

We now prove

Theorem 4.7: For any semi-ideal I of S, there exists a semilattice S' with $0'$ and a homomorphism $\theta: S \to S'$ such that $Ker\theta = I^*$.

Proof: Let I be a semi-ideal of S. Define $S' = \{f \mid f: I \to I$ is a homomorphism$\}$. Then S' is a $\wedge$ - semilattice where $(f \wedge g)(x) = f(x) \wedge g(x)$ for all $x \in I$ and $f, g \in S'$ with zero mapping as the zero element. For any $a \in S$ define $\psi_a: I \to I$ by $\psi_a(x) = a \wedge x$ for all $x \in I$. Obviously $\psi_a \in S'$ any $a \in S$. Further for any $x \in I$, $\psi_{a\wedge b}(x) = (\psi_a \wedge \psi_b)(x)$. This shows that $\psi_{a\wedge b} = \psi_a \wedge \psi_b$. Define $\theta: S \to S'$ by $\theta(a) = \psi_a$ for all $a \in S$. Then $\theta(a \wedge b) = \psi_{a\wedge b} = \psi_a \wedge \psi_b = \theta(a) \wedge \theta(b)$ for $a, b \in S$. Also $\theta(0) = \psi_0$, where ψ_0 is the zero element of S'. Thus θ is a homomorphism. Now $a \in Ker\theta \Leftrightarrow \theta(a) = \psi_0 \Leftrightarrow \psi_a = \psi_0 \Leftrightarrow \psi_a(x) = \psi_0(x)$ for all $x \in I$
$\Leftrightarrow a \wedge x = 0$ for all $x \in I \Leftrightarrow a \in I^*$. Therefore $Ker\theta = I^*$. □

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ANALYTIC CONTINUATION OF COMPLEX FUNCTIONS

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Abstract:

In this chapter we will to study about analytic continuation of different complex function first we will define somerelated definitions with illustrative examples. Analytic continuation is the addition of the domain of a given analytic function in the complex plane, to a larger domain of the complex plane this process has been utilized in many other areas of mathematics and has given mathematicians new insight into some of the world's hardest problems. This chapter also covers more general form of continuation.

Keywords: Analytic function, function element, Domain, common domain, chains of domains, Continuation.

Introduction:

Analytic continuation deals with the problem of properly reconsidering an analytic function so as to extend its domain of analycity. In other word if $f_1(z)$ be an analytic function in domain D_1, then there exist an analytic function $f_2(z)$ in different domain D_2 such that for $z \epsilon D_1 \cap D_2, f_1(z) = f_2(z)$, in this process we find that this type extension for many function not possible, but whenever this type of extension is possible it is unique

Funtion element:

A function element is an order pair set{f(z), D},in which D is a domain and f(z) is a single-valued analytic function, is known as function element. Two function element $\{f_1(z), D_1\}$ and $\{f_2(z), D_2\}$ are said to be equal if $D_1 = D_2, f_1(z) = f_2(z)$.

Definition:

Let $\{f_1(z), D_1\}$ and $\{f_2(z), D_2\}$ be the two function elements. Each function element is said to be a analytic continuation of the other if and only if $D_1 \cap D_2 \neq \emptyset$ and $f_1(z) = f_2(z)\ \forall\ z \epsilon D_1 \cap D_2$. Symbolically it is denoted by $\{f_1(z), D_1\} \sim \{f_2(z), D_2\}$.

Let $f_1(z)$ be the analytic function in domain D_1 and $f_2(z)$ be the analytic function in domain D_2 .Domain D_{12} be the common domain ofdomain D_1 and D_2. If $f_1(z) = f_2(z)$ in domain D_{12} , then $f_2(z)$ be the analytic continuation of $f_1(z)$ from D_1 into D_2 in common domain D_{12}. It also we can say direct analytic continuation.

Complete analytic function and natural boundary:

Let f(z) be analytic in domain D also let $f_1(z), f_2(z), f_3(z), \ldots\ldots, f_n(z)$ analytic continuation in domain $D_1, D_2, D_3, \ldots\ldots\ldots\ldots. D_n$ respectively. If there is a function $G(z)$ such that

$$G(z) = \begin{cases} f(z) \text{ in } D \\ f_1(z) \text{ in } D_1 \\ f_2(z) \text{ in } D_2 \\ \ldots.. \\ f_n(z) \text{ in } D_n \end{cases}$$

Then $G(z)$ is said to be complete analytic function in the domain wider than D, $D_1, D_2, D_3, \ldots\ldots\ldots, D_n$.

In this analytic continuation process we may arrive at a closed curve beyond which it is not possible to take analytic continuation. Such type closed curve is called natural boundary of the complete analytic function $G(z)$. A point outside these domain D, $D_1, D_2, D_3, \ldots\ldots\ldots, D_n$ is called singularities of the complete analytic function $G(z)$.

Theorem (1):

If a function analytic in domain D, vanishes over a part of D, then f(z) vanishes at each point of D.

Proof:

Let f(z) is vanishes over a domain D_0 is a part of D, then we are to show that it will be analytic every point of remaining part $D - D_0$ of domain D. We prove it by contradiction method if possible let there be a point p in

$D - D_0$ at which $f(p) \neq 0$. Take any point q in the part D_0 and join q to p by a curve C lying entirely in D clearly $f(q) = 0$.

Since f(z) is analytic in D, it is also continuous at point p; therefore there are points near p where f(z) does not vanish. Then there is a point r on the arc qp such that near r, f(z) vanishes points on the arc on one side and not vanish on other side, implying that f(z) is not continuous at point r. But this contradicts to our assumption that f(z)is analytic everywhere in D. Hence supposition is wrong (i.e. point p in $D - D_0$ at which f(p)=0).

Corollary:

If two functions, which are analytic in a domain D, coincide in a part of D, they coincide in the whole domain D.

Proof:

Let If $f_1(z)$ and $f_2(z)$be the two analytic functionin domain D and suppose D_0 is a part of D, in which If $f_1(z)$ and $f_2(z)$ coincide i.e.

$$f_1(z) = f_2(z) \ \forall z \epsilon D_0$$

Or $$f_1(z) - f_2(z) = 0 \ \forall z \epsilon D_0$$

Or $$f(z) = 0 \forall z \epsilon D_0$$

$$\Rightarrow f(z) = f_1(z) - f_2(z) = 0 \ \forall z \epsilon D_0$$

Since difference of two analytic function is also analytic function, therefore f(z) will be analytic in domain D and it will be vanish in the part D_0 of D. Therefore by the above theorem (If a function analytic in domain D, vanishes over a part of D, then f(z) vanishes at each point of D)f(z)will be vanish in whole domain D.

i. e. $$f(z) = f_1(z) - f_2(z) = 0 \ \forall z \epsilon D$$

Or $$f_1(z) = f_2(z) \forall z \epsilon D$$

Hence $f_1(z)$ and $f_2(z)$ are in whole domain D.

Theorem:

Let a function f(z)analytic in domain D. If f(z)vanishes over an arc L lying entirely in the domain D, then f(z)vanishes at each point of D.

Proof:

Let ζ be a point on arc L. Now taking ζ as the centre draw asmall circle such that which is totally lying entirely in domain D.

Since f(z) analytic at ζ, therefore f(z) vanishes everywhere within this small circle which is a part of D. Hence f(z) must vanish in the whole domain D.

Theorem:

There cannot be more than one analytic continuation of an analytic function $f_1(z)$ into the same domain.

Proof:

Let D_1 and D_2 be the two domain and D_{12} be the common domain of D_2 in D_1. Also let $f_1(z)$ be the analytic function in domain D_1 and $F_1(z)$ and $F_2(z)$ be the two analytic continuation of single analytic function $f_1(z)$ in Domain D_{12}.

Now we will have to prove $F_1(z) = F_2(z)$ In domain D_2.

Now by the definition of analytic continuation $f_1(z) = F_1(z) \forall z \epsilon D_{12}$..........(1)

(as $F_1(z)$ is the analytic continuation of $f_1(z)$)

And $F_1(z)$ be the analytic function in domain D_2 .

Similarly $f_1(z) = F_2(z) \forall z \epsilon D_{12}$.............(2)

(as $F_2(z)$ is the analytic continuation of $f_1(z)$)

And $F_2(z)$ be the analytic function in domain D_2

Now by equation(1) and equation (2), we have

$$f_1(z) = F_1(z) = F_2(z) \forall z \epsilon D_{12}$$

So we can say that and $F_1(z) = F_2(z)\ \forall z \epsilon D_{12}$

Or $F_1(z) - F_2(z) = 0\ \forall z \epsilon D_{12}$

Or $(F_1 - F_2)(z) = 0\ \forall z \epsilon D_{12}$

Since $F_1(z)$ and $F_2(z)$ both are analytic in D_2, so there difference $(F_1 - F_2)(z)$ is also analytic in domain D_2.

Clearly $(F_1 - F_2)(z)$ becomes zero in D_{12} part of D_2

$(F_1 - F_2)(z) = 0\ \forall z \epsilon D_2$

$$\Rightarrow F_1(z) = F_2(z) \forall z \epsilon D_2$$

Analytic continuation by a power series:

Let the Taylor's expansion of analytic function with respect to function z_1 be $f_1(z) = \sum_{n=0} a_n (z - z_1)^n$ ------- (1)

Which is convergent in circle $C_1: |z - z_1| = r_1 = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$ ----- (2)

Now draw a contour L from z_1 and consider analytic continuation of the function along L as follows:

Choose point z_2 such that arc z_1z_2 lie in C_1.

Now by equation (1) $f_1(z_2) = \sum_{n=0} a_n (z_2 - z_1)^n$

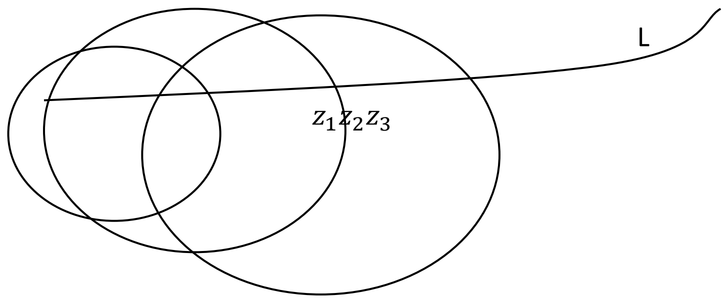
Therefore $f_1'(z_2) = \sum_{n=0} a_n n (z_2 - z_1)^{n-1}$

$$f_1''(z_2) = \sum_{n=0} a_n n(n-1)(z_2 - z_1)^{n-2}$$

---------------- and so on

Suppose that $f_2(z) = \sum_{n=0}^{\square} b_n(z - z_2)^n$ ---------- (3)

Where $b_n = \frac{{f_1}^n(z_2)}{n!}$ ------- (4)



Now the Power series (3) will be convergent in circle

$C_2: |z - z_2| = r_2 = \lim_{n\to\infty} (b_n)^{\frac{1}{n}}$ ---------- (5)

Again $f_1(z) = f_2(z)\ \forall\ z\epsilon$ in common domain of $C_1$and C_2 and $f_2(z)$will be the analytic continuation of $f_1(z)$from $C_1 \to C_2$.

Again on L choose point $z_3$such that arc$z_2z_3$lie entirely in C_2, then on differentiating (3), we can get

$$g(z_3), g'(z_3), g''(z_3), - - - -, g^n(z_3)$$

Now let $f_3(z) = \sum_{n=0}^{\square} c_n(z - z_2)^n$ -------------- (6)

Where $c_n = \frac{g^n(z_3)}{n!}$ ------------- (7)

Again the Power series (6) will be convergent in circle

$C_3: |z - z_3| = r_3 = \lim_{n\to\infty} (c_n)^{\frac{1}{n}}$

Now since $f_2(z) = f_3(z)\ \forall\ z\epsilon$ in common domain of $C_2$and C_3 so $f_3(z)$will be the analytic continuation of $f_2(z)$ from$C_2 \to C_3$.

On repeating this process function $f_1(z)$

Conclusion:

This chapter has covered just enough about analytic continuation of complex function in analytic continuation process we analyze continuity of analytic function in common closed domain analytic continuation along the any arc and found there cannot be more than be more than one analytic continuation of an analytic function into the same domain. We have presented it in a pedagogical way, in order to allow researchers to take full advantage of the methodology. Moreover, knowledge about the typical features of the considered spectra helps in choosing the rate at which appropriate operators are preferentially called, thus hybridizing a Genetic Algorithm with standard iterative methods, it is possible to outperform the phase retrieval capabilities of the algorithms used as memes to assist the genetic stochastic search

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STUDY OF TIME DEPENDENT SCHRODINGER'S WAVE EQUATION AND ITS DERIVATION

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Abstract:

In the present study the concept to learners in this field and further to improve the literature for knowledge of students which help to better understanding to mathematical formulation used which helps in understanding to derive it.It is shown that time dependent Schrodinger's Wave Equation (TDSE) may be derived using wave mechanics, time dependent equation,in classical.

Keywords: Simple harmonic motion, Schrödinger's Wave Equation, The wave function Ψ(x, y, z, t)

Introduction:

In our study,Schrödinger's Wave Equation is a mathematical expressionlike a differential equations that describes the behaviours of de-Broglie's matter wave associated with moving particle.Schrödingerproposed that a free particle is a similar to the motion of a simple harmonic progressive plane wave constant amplitude. Schrödingerassumed a variable Ψ called wave function for matter waves of a free particle. The wave function Ψ is a function of space variables (x, y, z) and time (t).

Derivation:

The time-dependent Schrödinger Wave Equation derivation is provided here so that students can learn the concept more effectively

The wave function $\Psi(x, t) = A\, e^{i(kx-\omega t)}$ represents a valid solution to the Schrödinger equation. The wave function is referred to as the free wave function as it represents a particle experiencing zero net force (at constant V).

The equation for wave function Ψ along positive X-direction is given by;

$\Psi = Ae^{-i\omega(t-x/v)}$

$$\Psi = Ae^{-i(\omega t - \omega x/v)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Where Ψ is a function of (x,t)

But from according to Plank's hypothesis

$$E = h\nu = \frac{h}{2\pi} \quad (\text{Since } \nu = \frac{\omega}{2\pi} \text{ and } \hbar = \frac{h}{2\pi})$$

$$E = h\frac{\omega}{2\pi} = \hbar\omega$$

$$\text{Thus } \omega = \frac{E}{\hbar} \quad \ldots\ldots\ldots\ldots\ldots(2)$$

According to de-Broglie hypothesis

$$P = \frac{h}{\lambda}$$

On multiplying and divide by 2π in above equation

$$P = \frac{h}{2\pi}.\frac{2\pi}{\lambda} = \hbar k \quad (\text{Since } k = \frac{2\pi}{\lambda})$$

$$k = \frac{P}{\hbar}$$

Now the wave velocity v is given by

$v = \nu\lambda$

On multiplying and divided by 2π

$$v = 2\pi\nu . \frac{\lambda}{2\pi} = \frac{\omega}{k} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

On putting the values of ω and v from equation (2) and (3) in equation(1)

$$\Psi = Ae^{-i(\frac{E}{\hbar}t - \omega x\frac{k}{\omega})}$$

On putting the value of k from above

$$\Psi = Ae^{-i(\frac{E}{\hbar}t - x\frac{p}{\hbar})}$$

$$\Psi = Ae^{-\frac{i}{\hbar}(Et - px)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Equation (4) represents the matter waves for a freeparticle of total energy E and momentum p moving along positive X-direction.

To find the equation of motion under the action of force, so the differentiating twice with respect to x, we get

$$\frac{\partial\Psi}{\partial x} = A(-\frac{i}{\hbar}).(-p)\, e^{-\frac{i}{\hbar}(Et - px)} = (\frac{ip}{\hbar})\Psi$$

$$\frac{\partial^2\Psi}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial\Psi}{\partial x}) = \frac{\partial}{\partial x}(\frac{ip}{\hbar})\Psi = \frac{ip}{\hbar}\frac{\partial\Psi}{\partial x} = \frac{ip}{\hbar}(\frac{ip}{\hbar})\Psi = i^2\frac{p^2}{\hbar^2}\Psi$$

$$\frac{\partial^2\Psi}{\partial x^2} = -\frac{p^2}{\hbar^2}\Psi \quad (\text{Since } i^2 = -1)$$

$$p^2\Psi = -\hbar^2\frac{\partial^2\Psi}{\partial x^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Now differentiating equation(4) with respect to time, we get

$$\frac{\partial \Psi}{\partial t} = A\left(-\frac{i}{\hbar}\right)E\, e^{-\frac{i}{\hbar}(Et - px)} \quad = \left(-\frac{iE}{\hbar}\right)\Psi$$

$$E\Psi = i\,\hbar\,\frac{\partial \Psi}{\partial t} \text{.......................(6) (Since } i^2 = -1)$$

We know that total energy E= K.E. + P.E.

$$E = \frac{p^2}{2m} + V$$

On multiplying by Ψ both side in above equation, we get

$$E\Psi = \frac{p^2}{2m}\,\Psi + V\,\Psi \quad \text{.......................................(7)}$$

On putting the value of $p^2\Psi$ and EΨ from equation (5) and (6) in equation (7),we get

$$i\,\hbar\,\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + + V\,\Psi \text{.....................................(8)}$$

The equation (8) is the time dependent equation for a particle in one dimension.

Now in three dimensional time dependent Schrodinger equation is given by

$$i\,\hbar\,\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2}\right) + V\,\Psi$$

$$\text{But,}\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} = \nabla^2\,\Psi \text{ and } \Psi = f(x,y,z,t)$$

$$i\,\hbar\,\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\,\Psi + V\,\Psi \text{............................(9)}$$

Thus, the time dependent Schrödinger equation, TDSE, can be derived from the wave mechanics considering the equations for a particle describing S.H.M.

Physical significance of wave function Ψ:

The most satisfactory interpretation by a wave function Ψassociated with the moving particle was given by Max Born in 1926.

Each moving particle is a wave function Ψ which is a function of space variables (x, y, z) and time‘t’.

According to Max Born, square of the magnitude of the wave function i.e. $\|\Psi\|^2$ or $\Psi^*\Psi$ evaluated in a particular region represent the probability of finding the particle at that position and at that time.

Where, Ψ^* = complex conjugate of Ψ

Hence, $\|\Psi\|^2$ is known as Probability density.

Let the complex wave function Ψ with real and imaginary parts is given by

Ψ = x+ i y Where, x and y are real functions.

$\Psi^* = x - i\, y$

Now, $\Psi^* \Psi = (x - i\, y)(x + i\, y) = x^2 + y^2$ = Real

Hence $\Psi^* \Psi$ is always a positive real quantity.

So the probability density, $\|\Psi\|^2$ which must be appositive real quantity is given by product $\Psi^* \Psi$.

The probability of finding a particle is certainly found somewhere in the space, the integral of $\|\Psi\|^2$ dV over all the space must be equal to unity.

$\int_{-\infty}^{\infty} |\Psi|^2 dV = 1$ or $\int_{-\infty}^{\infty} \Psi * \Psi\, dV = 1$

Where dV= dx.dy.dz

If $\int_{-\infty}^{\infty} |\Psi|^2 dV = 0$

This show that particle does not exist.

Conclusions:

In this expression, Schrödinger's Wave Equation is plays a very important role to find permitted energy levels of atoms. The solution of time dependent equation of a wave that describes the quantum aspects of a system. In quantum mechanics thisarticle have an insight into SE and varieties of way to derive.

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INTRODUCTION TO SAMPLING THEORY

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Statistics is the science of data. Data are the numerical values containing some information. Statistical tools can be used on a data set to draw statistical inferences. These statistical inferences are in turn used for various purposes. For example, government uses such data for policy formulation for the welfare of the people, marketing companies use the data from consumer surveys to improve the company and to provide better services to the customer, etc. Such data is obtained through sample surveys. Sample surveys are conducted throughout the world by governmental as well as non- governmental agencies. For example, "National Sample Survey Organization (NSSO)" conducts surveys in India, "Statistics Canada" conducts surveys in Canada, agencies of United Nations like "World Health Organization (WHO), "Food and Agricultural Organization (FAO)" etc. conduct surveys in different countries.

Sampling theory provides the tools and techniques for data collection keeping in mind the objectives to be fulfilled and nature of population.

There are two ways of obtaining the information

1. **Sample surveys**
2. **Complete enumeration or census**

Sample surveys collect information on a fraction of total population whereas census collects information on whole population. Some surveys e.g., economic surveys, agricultural surveys etc. are conducted regularly. Some surveys are need based and are conducted when some need arises, e.g., consumer satisfaction surveys at a newly opened shopping mall to see the satisfaction level with the amenities provided in the mall.

Sampling unit:

An element or a group of elements on which the observations can be taken is called a sampling unit. The objective of the survey helps in determining the definition of sampling unit. For example, if the objective is to determine the total income of all the persons in the household, then the sampling unit is household. If the objective is to determine the income of any particular person in the household, then the sampling unit is the income of the particular person in the household. So the definition of sampling unit depends and varies as per the objective of the survey. Similarly, in another example, if the objective is to study the blood sugar level, then the sampling unit is the value of blood sugar level of a person. On the other hand, if the objective is to study the health conditions, then the sampling unit is the person on whom the readings on the blood sugar level, blood pressure and other factors will be obtained. These values will together classify the person as healthy or unhealthy.

Population:

Collection of all the sampling units in a given region at a particular point of time or a particular period is called the population. For example, if the medical facilities in a hospital are to be surveyed through the patients, then the total number of patients registered in the hospital during the time period of survey wills the population. Similarly, if the production of wheat in a district is to be studied, then all the fields cultivating wheat in that district will be constitute the population. The total number of sampling units in the population is the population size, denoted generally by N. The population size can be finite or infinite (N is large).

Census:

The complete count of population is called census. The observations on all the sampling units in the population are collected in the census. For example, in India, the census is conducted at every tenth year in which observations on all the persons staying in India is collected.

Sample:

One or more sampling units are selected from the population according to some specified procedure. A sample consists only of a portion of the population units. Such a collection of units is called the sample.

In the context of sample surveys, a collection of units like households, people, cities, countries etc. is called a finite population.

A census is a 100% sample and it is a complete count of the population.

Representative sample:

When the entire salient features of the population are present in the sample, then it is called a representative sample, It goes without saying that every sample is considered as a representative sample. For example, if a population has 30% males and 70% females, then we also expect the sample to have nearly 30% males and 70% females.

In another example, if we take out a handful of wheat from a 100 Kg. bag of wheat, we expect the same quality of wheat in hand as inside the bag. Similarly, it is expected that a drop of blood will give the same information as all the blood in the body.

Sampling frame:

The list of all the units of the population to be surveyed constitutes the sampling frame. All the sampling units in the sampling frame have identification particulars. For example, all the students in a particular university listed along with their roll numbers constitute the sampling frame. Similarly, the list of households with the name of head of family or house address constitutes the sampling frame. In another example, the residents of a city area may be listed in more than one frame - as per automobile registration as well as the listing in the telephone directory.

Ways to ensure representativeness:

There are two possible ways to ensure that the selected sample is representative.

1. **Random sample or probability sample:**

The selection of units in the sample from a population is governed by the laws of chance or probability. The probability of selection of a unit can be equal as well as unequal.

2. **Non-random sample or purposive sample:**

The selection of units in the sample from population is not governed by the probability laws.

For example, the units are selected on the basis of personal judgment of the surveyor. The persons volunteering to take some medical test or to drink a new type of coffee also constitute the sample on non-random laws.

Another type of sampling is Quota Sampling. The survey in this case is continued until a pre-determined number of units with the characteristic under study are picked up. For example, in order to conduct an experiment for rare type of disease, the survey is continued till the required number of patients with the disease is collected.

Advantages of sampling over complete enumeration:

1. Reduced cost and enlarged scope:

Sampling involves the collection of data on smaller number of units in comparison to the complete enumeration, so the cost involved in the collection of information is reduced. Further, additional information can be obtained at little cost in comparison to conducting another separate survey. For example, when an interviewer is collecting information on health conditions, then he/she can also ask some questions on health practices. This will provide additional information on health practices and the cost involved will be much less than conducting an entirely new survey on health practices.

2. Organization of work:

It is easier to manage the organization of collection of smaller number of units than all the units in a census. For example, in order to draw a representative sample from a state, it is easier to manage to draw small samples from every city than drawing the sample from the whole state at a time. This ultimately results in more accuracy in the statistical inferences because better organization provides better data and in turn, improved statistical inferences are obtained.

3. Greater accuracy:

The persons involved in the collection of data are trained personals. They can collect the data more accurately if they have to collect smaller number of units than large number of units.

4. Urgent information required:

The data from a sample can be quickly summarized.

For example, the forecasting of the crop production can be done quickly on the basis of a sample of data than collecting first all the observation.

5. Feasibility:

Conducting the experiment on smaller number of units, particularly when the units are destroyed, is more feasible. For example, in determining the life of bulbs, it is more feasible to fuse minimum number of bulbs. Similarly, in any medical experiment, it is more feasible to use less number of animals.

Type of surveys:

There are various types of surveys which are conducted on the basis of the objectives to be fulfilled.

1. Demographic surveys:

These surveys are conducted to collect the demographic data, e.g., household surveys, family size, number of males in families, etc. Such surveys are useful in the policy formulation for any city, state or country for the welfare of the people.

2. Educational surveys:

These surveys are conducted to collect the educational data, e.g., how many children go to school, how many persons are graduate, etc. Such surveys are conducted to examine the educational programs in schools and colleges. Generally, schools are selected first and then the students from each school constitue the sample.

3. Economic surveys:

These surveys are conducted to collect the economic data, e.g., data related to export and import of goods, industrial production, consumer expenditure etc. Such data is helpful in constructing the indices indicating the growth in a particular sector of economy or even the overall economic growth of the country.

4. Employment surveys:

These surveys are conducted to collect the employment related data, e.g., employment rate, labour conditions, wages, etc. in a city, state or country. Such data helps in constructing various indices to know the employment conditions among the people.

5. Health and nutrition surveys:

These surveys are conducted to collect the data related to health and nutrition issues, e.g., number of visits to doctors, food given to children, nutritional value etc. Such surveys are conducted in cities, states as well as countries by the national and international organizations like UNICEF, WHO etc.

6. Agricultural surveys:

These surveys are conducted to collect the agriculture related data to estimate, e.g., the acreage and production of crops, livestock numbers, use of fertilizers, use of pesticides and other related topics. The government bases its planning related to the food issues for the people based on such surveys.

7. Marketing surveys:

These surveys are conducted to collect the data related to marketing. They are conducted by major companies, manufacturers or those who provide services to consumer etc. Such data is used for knowing the satisfaction and opinion of consumers as well as in developing the sales, purchase and promotional activities etc.

8. Election surveys:

These surveys are conducted to study the outcome of an election or a poll. For example, such polls are conducted in democratic countries to have the opinions of people about any candidate who is contesting the election.

9. Public polls and surveys:

These surveys are conducted to collect the public opinion on any particular issue. Such surveys are generally conducted by the news media and the agencies which conduct polls and surveys on the current topics of interest to public.

10. Campus surveys:

These surveys are conducted on the students of any educational institution to study about the educational programs, living facilities, dining facilities, sports activities, etc.

Principal steps in a sample survey:

The broad steps to conduct any sample surveys are as follows:

1. **Objective of the survey:**

The objective of the survey has to be clearly defined and well understood by the person planning to conduct it. It is expected from the statistician to be well versed with the issues to be addressed in consultation with the person who wants to get the survey conducted. In complex surveys, sometimes the objective is forgotten and data is collected on those issues which are far away from the objectives.

2. **Population to be sampled:**

Based on the objectives of the survey, decide the population from which the information can be obtained. For example, population of farmers is to be sampled for an agricultural survey whereas the population of patients has to be sampled for determining the medical facilities in a hospital.

3. **Data to be collected:**

It is important to decide that which data is relevant for fulfilling the objectives of the survey and to note that no essential data is omitted. Sometimes, too many questions are asked and some of their outcomes are never utilized. This lowers the quality of the responses and in turn results in lower efficiency in the statistical inferences.

4. **Degree of precision required:**

The results of any sample survey are always subjected to some uncertainty. Such uncertainty can be reduced by taking larger samples or using superior instruments. This involves more cost and more time. So it is very important to decide about the required degree of precision in the data. This needs to be conveyed to the surveyor also.

5. **Method of measurement:**

The choice of measuring instrument and the method to measure the data from the population needs to be specified clearly. For example, the data has to be collected through interview, questionnaire, personal visit, combination of any of these approaches, etc. The forms in which the data is to be recorded so that the data can be transferred to mechanical equipment for easily creating the data summary etc. is also needed to be prepared accordingly.

6. **The frame:**

The sampling frame has to be clearly specified. The population is divided into sampling units such that the units cover the whole population and every sampling unit is tagged with identification. The list of all sampling units is called the frame. The frame must cover the whole population and the units must not overlap each other in the sense that every element in the population must belong to one and only one unit. For example, the sampling unit can be an individual member in the family or the whole family.

7. Selection of sample:

The size of the sample needs to be specified for the given sampling plan. This helps in determining and comparing the relative cost and time of different sampling plans. The method and plan adopted for drawing a representative sample should also be detailed.

8. The Pre-test:

It is advised to try the questionnaire and field methods on a small scale. This may reveal some troubles and problems beforehand which the surveyor may face in the field in large scale surveys.

9. Organization of the field work:

How to conduct the survey, how to handle business administrative issues, providing proper training to surveyors, procedures, plans for handling the non-response and missing observations etc. are some of the issues which need to be addressed for organizing the survey work in the fields. The procedure for early checking of the quality of return should be prescribed. It should be clarified how to handle the situation when the respondent is not available.

10. Summary and analysis of data:

It is to be noted that based on the objectives of the data, the suitable statistical tool is decided which can answer the relevant questions. In order to use the statistical tool, a valid data set is required and this dictates the choice of responses to be obtained for the questions in the questionnaire, e.g., the data has to be qualitative, quantitative, nominal, ordinal etc. After getting the completed questionnaire back, it needs to be edited to amend the recording errors and delete the erroneous data. The tabulating procedures, methods of estimation and tolerable amount of error in the estimation need to be decided before the start of survey. Different methods of estimation may be available to get the answer of the same query from the same data set. So the data needs to be collected which is compatible with the chosen estimation procedure.

11. Information gained for future surveys:

The completed surveys work as guide for improved sample surveys in future. Beside this they also supply various types of prior information required to use various statistical tools, e.g., mean, variance, nature of variability, cost involved etc. Any completed sample survey acts as a potential guide for the surveys to be conducted in the future. It is generally seen that the things always do not go in the same way in any complex survey as planned earlier. Such precautions and alerts help in avoiding the mistakes in the execution of future surveys.

Variability control in sample surveys:

The variability control is an important issue in any statistical analysis. A general objective is to draw statistical inferences with minimum variability. There are various types of sampling schemes which are adopted in different conditions. These schemes help in controlling the variability at different stages. Such sampling schemes can be classified in the following way.

1. **Before selection of sampling units**
   - Stratified sampling
   - Cluster sampling
   - Two stage sampling
   - Double sampling etc.

2. **At the time of selection of sampling units**
   - Systematic sampling
   - Varying probability sampling

3. **After the selection of sampling units**
   - Ratio method of estimation
   - Regression method of estimation

Note that the ratio and regression methods are the methods of estimation and not the methods of drawing samples.

Methods of data collection:

There are various ways of data collection. Some of them are as follows:

1. **Physical observations and measurements:**

The surveyor contacts the respondent personally through the meeting. He observes the sampling unit and records the data. The surveyor can always use his prior experience to collect the data in a better way. For example, a young man telling his age as 60 years can easily be observed and corrected by the surveyor.

2. Personal interview:

The surveyor is supplied with a well prepared questionnaire. The surveyor goes to the respondents and asks the same questions mentioned in the questionnaire. The data in the questionnaire is then filled up accordingly based on the responses from the respondents.

3. Mail enquiry:

The well prepared questionnaire is sent to the respondents through postal mail, e-mail, etc. The respondents are requested to fill up the questionnaires and send it back. In case of postal mail, many times the questionnaires are accompanied by a self addressed envelope with postage stamps to avoid any non-response due to the cost of postage.

4. Web based enquiry:

The survey is conducted online through internet based web pages. There are various websites which provide such facility. The questionnaires are to be in their formats and the link is sent to the respondents through email. By clicking on the link, the respondent is brought to the concerned website and the answers are to be given online. These answers are recorded and responses as well as their statistics are sent to the surveyor. The respondents should have internet connection to support the data collection with this procedure.

5. Registration:

The respondent is required to register the data at some designated place. For example, the number of births and deaths along with the details provided by the family members are recorded at city municipal office which are provided by the family members.

6. Transcription from records:

The sample of data is collected from the already recorded information. For example, the details of the number of persons in different families or number of births/deaths in a city can be obtained from the city municipal office directly.

The methods in (1) to (5) provide primary data which means collecting the data directly from the source. The method in (6) provides the secondary data which means getting the data from the primary sources.

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OPEN MAPPING THEORM: AN APPLICATION OF BAIRE'S CATEGORY THEOREM

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Abstract:

In this chapter, our goal is to expound the importance of Baire's Category Theorem in mathematics. In achieve this goal; we study a very famous result of functional analysis, namely, the Open Mapping Theorem. The Open Mapping Theorem and Baire Category Theorem are foundational results in functional analysis, particularly related to the study of Banach spaces (complete normed vector spaces). The Open Mapping Theorem states that a surjective bounded linear operator between Banach spaces is an open map, meaning it maps open sets to open sets. This theorem's proof heavily relies on the Baire Category Theorem, which states that a complete metric space cannot be expressed as a countable union of nowhere dense sets.

1. Introduction:

In mathematics, the Baire's category theorem is an important tool in the study of complete spaces, such as Banach spaces and Hilbert spaces that arise in topology and functional analysis. In functional analysis, two of the most powerful theorems, the open mapping theorem and uniform boundedness principle are direct consequences of the Baire's category theorem. The principle of uniform boundedness by S. Banach and Hilbert Steinhaus (1927) is of great importance. In fact, throughout analysis there are many instances of results related to this principle, the earliest being an investigation by H. Lebesgue (1909). The principle of uniform boundedness is often regarded as one of the cornerstones of functional analysis in normed spaces, the others being the Hahn –Banach theorem, the open mapping theorem and the closed graph theorem. Unlike the Hahn – Banach theorem, other three of these four theorems require completeness. Indeed, they characterize some of the most important properties of Banach spaces which normed spaces in general may not have. But we obtain all three theorems from a common source, which is

Baires category theorem. From the Baire's category theorem, we can derive the principle of uniform boundedness and the open mapping theorem. On the other hand, the closed graph theorem is an application of the open mapping theorem.

2. Preliminaries:

The open mapping theorem will be concerned with open mappings. These are mappings such that the image of every open set is an open set (definition below). It is well known that open mappings are of general interest. As in the uniform boundedness theorem we again need completeness and present theorem exhibits another reason why Banach spaces are more satisfactory than incomplete normed spaces. The theorem also gives conditions under which the inverse of a bounded linear operator is bounded. The proof of the open mapping theorem is based on Baire's category theorem. To establish this result, we need a lemma and the proof of this lemma depends on Baire's category theorem.

Lemma 2.1: Let X and Y be two Banach spaces, and let $T: X \to Y$ be an onto continuous linear operator. If zero is an interior point of a subset A of X , then zero is also an interior point of $T(A)$.

Proof: Set $V = \{x \in X : \|x\| \leq 1\}$ and observe that $rV = \{rx : x \in V\}$ is the closed ball with center at zero and radius r. Since zero is assumed to be an interior point of A, there exists $r > 0$ with $rV \subseteq A$. By the linearity of T we must have $T(rV) = rT(V) \subseteq T(A)$. Hence to establish the result it is enough to show that zero is an interior point of $T(V)$.

Clearly, $X = \bigcup_{n=1}^{\infty} nV$ holds and since T is an onto linear operator, $Y = \bigcup_{n=1}^{\infty} nT(V)$ also holds. By Baire's category theorem, there exists some K such that $\overline{kT(V)}$ has a nonempty interior. Since $\overline{kT(V)} = k\overline{T(V)}$, it follows that $\overline{T(V)}$ has an interior point. That is, there exists some $y_0 \in \overline{T(V)}$ and $r > 0$ such that $B(y_0, 2r) \subseteq \overline{T(V)}$,. Now if $y \in Y$ satisfies $\|y\| < 2r$, then $y - y_0 \in \overline{T(V)}$.

Therefore $y = (y - y_0) + y_0 \in \overline{T(V)} + \overline{T(V)} = 2\overline{T(V)}$,

where the last inclusion follows from the identity $V + V = 2V$. That is,

$$\{y \in Y : \|y\| < r\} \subseteq 2\overline{T(V)}.$$

By the linearity of T, it follows that, $\{y \in Y : \|y\| < r2^{-n}\} \subseteq 2^{-n}\overline{T(V)} = \overline{T(2^{-n}V)}$

holds for each n.

Now let $y \in Y$ be fixed such that $\|y\| < r2^{-1}$.. Since $y \in \overline{T(2^{-1}V)}$, there exists some $x_1 \in 2^{-1}V$ such that $\|y - T(x_1)\| < r2^{-2}$. Now proceed inductively.

Assume that x_n has been selected such that $x_n \in 2^{-n}V$ and $\left\| y - \sum_{i=1}^{n} T(x_i) \right\| < r2^{-n-1}$..

Clearly $y - \sum_{i=1}^{n} T(x_i) \in \overline{T(2^{-n-1})V}$ and so there exists some $x_{n+1} \in 2^{-n-1}V$ with $\left\| y - \sum_{i=1}^{n+1} T(x_i) \right\| < r2^{-n-2}$. Thus, a sequence $\{x_n\}$ is selected such that $\|x_n\| \le 2^{-n}$ and $\left\| y - \sum_{i=1}^{n} T(x_i) \right\| = \left\| y - T(\sum_{i=1}^{n} x_i) \right\| < 2r^{-n-1}$ holds for all n.

Next define $s_n = x_1 + x_2 + ... + x_n$ for each n, and note that $\|s_{n+p} - s_n\| = \left\| \sum_{i=n+1}^{n+p} x_i \right\| \le \sum_{i=n+1}^{n+p} \|x_i\| \le 2^{-n}$ for all n and p shows that $\{s_n\}$ is a Cauchy sequence .Since X is Banach, so $\{s_n\}$ is convergent. Let $x = \lim s_n$ in X. Then $\|x\| \le \sum_{n=1}^{\infty} \|x_n\| \le 1$ (i.e $x \in V$) and by the continuity and linearity of T, we get $T(x) = \lim_{n\to\infty} T(s_n) = \lim \sum_{i=1}^{n} T(x_i) = y$.

That is, $y \in T(V)$, and so $\{y \in Y : \|y\| < \frac{r}{2}\} \subseteq T(V)$.The proof of the lemma is now complete.

Definition 2.2: Let X and Y be metric spaces. Then $f : X \to Y$ is called an open mapping if $f(A)$ is open in Y whenever A is open in X.

3.Main Theorem (Open Mapping Theorem):

Let X and Y be two Banach spaces, and let $T : X \to Y$ be a bounded linear operator .If T is onto, then T is an open mapping.

Proof: Let O be an open subset of X and let $y \in T(O)$. Pick a point $x \in O$ such that $y = T(x)$, and note that $y - T(O) = T(x - O)$ holds. Now observe that zero is an interior point of $x - O$ hence, by lemma 2.1, zero is also an interior point of $y - T(O) = T(x - O)$. This implies that y is an interior of $T(O)$. Since y is arbitrary, $T(O)$ is an open set, and the proof of the theorem is complete.

Remark: In addition T is one to one, then T is a homeomorphism. Because if T is one to one, then T will be a bijection and hence T^{-1} is exists. By open mapping theorem T is open and hence T^{-1} is bounded as if M is open in X then $(T^{-1})^{-1}(M)=T(M)$ is open in Y. Thus if $T:X\to Y$, where X and Y are Banach spaces, is a bijection then T^{-1} is bounded and this result is known as **Bounded Inverse Theorem.**

Theorem 3.1. Let X be a linear vector space that is complete in each of the norms $\| \ \|_1$ and $\| \ \|_2$, and suppose that there is a constant c such that $\|x\|_1 \le c\|x\|_2$ for all $x\in X$. Then the norms are equivalent. That is, there is second constant c^1 such that $\|x\|_2 \le c^1\|x\|_1$ for all $x\in X$.

Proof: Consider the identity mapping $I:(X,\| \ \|_2)\to(X,\| \ \|_1)$. Clearly I is one to one and onto. Since $\|x\|_1 \le c\|x\|_2$ for all $x\in X$ so I is bounded. Again by open mapping theorem I is open. Hence I^{-1} is bounded. This implies that there exists some c^1 such that $\|x\|_2 \le c^1\|x\|_1$ for all $x\in X$.

Theorem 3.2: Let $T:X\to Y$ be a bounded linear operator where X and Y are Banach spaces. If T is bijective then there exists $a,b>0$ such that $a\|x\|\le\|Tx\|\le b\|x\|$ for all $x\in X$

Proof: Since T is bounded so there exists $b>0$ such that $\|Tx\|\le b\|x\|$ for all $.\,x\in X$ …….(1)

Since T is bijective, i.e. $T^{-1}:X\to Y$ exists and $TT^{-1}=I=T^{-1}T$.

Therefore by open mapping theorem, T^{-1} is bounded so there exists $\lambda>0$ such that

$$\|T^{-1}y\|\le\lambda\|y\| \text{ for all } y\in Y$$

$$\Rightarrow \|T^{-1}T(x)\|\le\lambda\|y\|$$

$$\Rightarrow \|x\|\le\lambda\|Tx\| \text{ for all } x\in X$$

$$\Rightarrow \frac{1}{\lambda}\|x\|\le\|Tx\|$$

$$\Rightarrow a\|x\|\le\|Tx\| \text{ where } \frac{1}{\lambda}=a \text{ ……(2)}$$

Therefore (1), (2) $\Rightarrow a\|x\|\le\|Tx\|\le b\|x\|$ for all $x\in X$

This completes the proof.

We know that $C[0,1]$, the collection of all real valued continuous functions defined on $[0,1]$ is a vector space over R. Now we define two norms in $C[0,1]$ as

$$\|f\|_\infty=\sup\{|f(x)|:x\in[0,1]\} \text{ and}$$

$$\|f\|_1 = \int_0^1 |f(x)|dx \text{ for } f \in C[0,1].$$

Theorem 3.3: $(C[0,1], \|f\|_\infty)$ is a Banach space.

Proof: See page 35, example 5.16 in [2].

Theorem 3.4: $(C[0,1], \| \ \|_1)$ is not a Banach space.

Proof: Define the identity operator $I : (C[0,1], \| \ \|_\infty) \to (C[0,1], \| \ \|_1)$

Then $\|I\| = \sup\{\|I(f)\|_1 : \|f\|_\infty = 1, f \in C[0,1]\}$

Now $\|I(f)\|_1 = \|f\|_1 = \int_0^1 |f(x)dx| \le \|f\|_\infty$ so $\|I\| \le 1$.

Also $f(x) = 1$ for all $x \in [0,1]$ is a continuous function and $\|f\|_\infty = 1$, so $\|I\| = 1$.

Hence I is bounded, i.e., continuous operator.

Again consider the identity operator $I^{-1} : (C[0,1], \| \ \|_1) \to (C[0,1], \| \ \|_\infty)$.

Define $f_n(x) = (n+1)x^n$ for each n

Then $$\|f_n\|_1 = \int_0^1 |(n+1)x^n| dx = \int_0^1 (n+1)x^n dx = 1$$

Also $\|f_n\|_\infty = n+1$ Now $\|I^{-1}\| = \sup\{\|I^{-1}(f)\|_\infty : \|f\|_1 = 1\}$

$$\ge \sup\{\|I^{-1}(f_n)\|_\infty\}$$

$$= \sup\{\|f_n\|_\infty\}$$

$$= \sup\{n+1 : n \in N\}$$

$$= \infty$$

Hence I^{-1} is bounded and therefore I is not open. But I is bounded, onto. Therefore by open mapping theorem either $(C[0,1], \|f\|_\infty)$ or $(C[0,1], \| \ \|_1)$ is not Banach. But by theorem 3.3 $(C[0,1], \|f\|_\infty)$ is a Banach space. Hence $(C[0,1], \| \ \|_1)$ is not Banach.

In connection with the openness property of a linear map, the following result is useful.

Proposition 3.5: Let T be a bounded linear map from a normed linear space X on to a normed linear space Y .Then T is open if and only if there is $\lambda > 0$ such that for each $y \in Y$, there is $x \in X$ where $T(x) = y$ and $\|x\| \le \lambda \|y\|$.

Remark: The above result enables us to obtain a partial converse of the open mapping theorem.

Theorem 3.6: Let X and Y be Banach spaces. Then the set of all surjective maps in $L(X,Y)$ is open in $L(X,Y)$.

Proof: Let T be a surjective map in $L(X,Y)$. Let $U \in L(X,Y)$ and $\|T-U\| < \frac{1}{2^k},$ where k is a real number with the property as in above proposition. We have prove that U is surjective.

Let $y \in Y$ and $\|y\| \le 1$, it follows from above proposition that there exists $x \in X$ such that $T(x) = y$ and $\|x\| \le k$. Let $y_1 = T(x) - U(x).$ Then $\|y_1\| \le \frac{1}{2}$ and there exists $x_1 \in X$ such that $T(x_1) = y_1$ and $\|x_1\| \le \frac{k}{2}$ .Let $y_2 = T(x_1) - U(x_1).$then $\|y_2\| \le \frac{1}{2^2}$. Continuing inductively, we can find x_n such that $T(x_n) = y_n$ and $y_{n+1} = T(x_n) - U(x_n)$

$$\|y_n\| \le \frac{1}{2^n}, \qquad \|x_n\| \le \frac{k}{2^n}$$

It then follows that $y = U(x) + U(x_1) + \dots\dots + U(x_n).$

Thus $U(z) = y$ where $z = \sum_{n=1}^{\infty} x_n$. This proves the theorem.

Now we move to another consequence of open mapping theorem. i.e. Open mapping theorem is used in Factor spaces. The results on factor spaces and direct sums represent important auxiliary tools for the investigation of linear and nonlinear operator equations.

But to show this, we need some definitions.

Let L be a linear subspace of the linear space X over K. For all $u, v \in X$, we define

$$u \equiv v(\text{mod} L) \text{ iff } u - v \in L \dots\dots\dots\dots\dots\dots(1)$$

This is an equivalence relation. In fact, for all $u, v, w, z \in X$ and $\alpha \in K$, we have the following

$$u \equiv v(\text{mod} L)$$

$$u \equiv v(\text{mod} L) \Rightarrow v \equiv u(\text{mod} L)$$

$$u \equiv v(\text{mod} L),\ v \equiv w(\text{mod} L) \Rightarrow u \equiv w(\text{mod} L)$$

This equivalence relation is compatible with the linear structure of L:

$$u \equiv v(\text{mod} L) \Rightarrow \alpha u = \alpha v(\text{mod} L) \dots\dots\dots\dots(2)$$

$$u \equiv w(\text{mod} L),\ v \equiv z(\text{mod} L) \Rightarrow u + v \equiv w + z(\text{mod} L)$$

Definition 3.7: The factor space X/L consists of all the equivalence classes $[u]$ with respect to (1), that is $v \in [u]$ iff $u \equiv v(\text{mod} L)$. Explicitly, this means that $[u] = u + L$

The elements v of the class $[u]$ are called the representatives of $[u]$. Obviously $[u] = [v] \Leftrightarrow$ $u \equiv v(\text{mod} L)$ ....(3)

If we introduce the linear operations $\alpha[u] = [\alpha u]$

$$[u] + [v] = [u + v] \quad \ldots\ldots\ldots\ldots\ldots(4)$$

the factor space X/L becomes a linear space. The operations in (4) are well defined namely; they are independent of the chosen representatives. This follows from (2) and (3). For example, if $[u] = [v]$ then $u \equiv v(\text{mod} L)$ and hence $\alpha[u] = [\alpha u](\text{mod} L)$ that is $[\alpha u] = [\alpha v]$.

In other words, the factor space X/L consists of all the different sets $u + L$ where $u \in X$ and linear operations on X/L are given through $(u + L) + (v + L) = (u + v) + L$ and $\alpha(u + L) = \alpha u + L$ which corresponds to the usual operations $A + B$ and αA for subsets A and B of linear spaces.

Proposition 3.8: Let L be a closed linear subspace of the normed space X over K. Then the following are true.

(1) The factor space X/L becomes a normed space over K with respect to the norm $\|[u]\| = \inf_{v \in [u]} \|v\|$

(2) If X is a Banach space then so is X/L

Since $[u] = u + L$ we get $\|[u]\| = dist(0, u + L) = dist(u, L)$

Definition 3.9: Let L be a linear subspace of the linear space X over K. Then the canonical mapping $\Pi : X \to X/L$ is defined through $\Pi(u) = [u]$ for all $u \in X$ where $[u] = u + L$

Proposition 3.10: If L is a closed linear subspace of the normed space X over K, then the canonical mapping $\Pi : X \to X/L$ is linear, continuous and surjective.

Proof: See page 187 in [10].

Remark: Let $A : X \to Y$ be a linear continuous operator, where X and Y are Banach spaces over K. We define the operator $[A] : X/N(A) \to R(A)$ ....(A) through $[A][u] = Au$

.

This definition is independent of the selected representatives. In fact let $[u]=[v]$. Then $u-v \in N(A)$ that is $A(u-v)=0$ and hence $Au = Av$

Proposition 3.11: Let the range $R(A)$ of the operator A be closed.

(1) The operator $[A]$ from (A) ids a linear homeomorphism.

(2) There exists a number $c > 0$ such that

$c..dist(u, N(A)) \leq \|Au\|$ for all $u \in X$

Proof (1) : The null space $N(A) = \{u \in X : Au = 0\}$ is closed. In fact, if $Au_n = 0$ and $u_n \to u$ as $n \to \infty$ then $Au = 0$. Thus $X/N(A)$ is Banach space. Obviously, the operator $[A]$ is linear. Since $\|[A][u]\| = \|Av\| \leq \|A\|\|v\|$ for all $v \in [u]$

We have $\|[A][u]\| \leq \|A\|\|[u]\|$ and thus $[A]$ is continuous.

Furthermore, the operator $[A]$ is bijective. Infact, if $[A][u] = 0$ then $u \in N(A)$ and hence $[u] = 0$.

Since $R(A)$ is closed linear subspace of the Banach space Y the range $R(A)$ is also Banach space. The Bounded inverse theorem tells us that the inverse operator $[A]^{-1} : R(A) \to X/N(A)$ is continuous.

(2) By (1) there is a constant $d > 0$ such that $\|[A]^{-1}[u]\| \leq d\|[u]\|$ for all $[u] \in X/N(A)$. Hence $\|[v]\| \leq d\|[A][v]\|$ for all $[v] \in X/N(A)$.

Hence (2) where $c = d^{-1}$

Now move to direct sum and projections- where Bounded inverse theorem is used.

Definition 3.12: Let X be linear space over K and let L_1 and L_2 be linear subspaces of X .

(1) We write $X = L_1 \oplus L_3$ iff each $u \in X$ allows the following unique representation:

$u = u_1 + u_3$ where $u_1 \in L_1$ and $u_2 \in L_2$ ....(B)

We say that X is the direct sum of L_1 and L_2 and that L_2 is an algebraic complement of L_1 in X.

(2) The operator $p : X \to X$ is called an algebraic projection iff p is linear and $p^2 = p$

(3) If X is normed space, then the operator $p: X \to X$ is called an continuous projection iff p is a continuous algebraic projection. Obviously $X = L_1 \oplus L_3$ iff $X = L_2 \oplus L_1$. Moreover, let $X = L_1 \oplus L_3$, then $u \in L_1 \cap L_2$ implies $u = 0$.

This follows from $u = u + 0 = 0 + u$ and from the uniqueness of the decomposition in (B).

Now if we consider the linear operator equation $Au = b$, $u \in X$. Then we have the following theorem—

Theorem 3.13: Supposed that the operator $A: X \to X$ is linear, where X and Y are linear spaces over K. Let L bee any fixed algebraic complement of the null space $N(A)$, namely L is a linear subspace of X such that $X = N(A) \oplus L$ ......(**)

Then the following statements are true.

(1) The restriction $A: L \to R(A)$ is linear and bijective. Hence codim $N(A) = \dim R(A)$

(2) In addition, suppose that X and Y are Banach spaces L and $R(A)$ are closed, and the operator $A: X \to Y$ is continuous. Then the operator from (c) is a linear homomorphism.

[Since $R(A) = A(X)$. The number $\dim R(A)$ is called the rank of A. We denote this as rank A =dim $R(A)$]

Proof: (1) It follows from $Au = 0$ with $u \in L$ that $u \in N(A) \cap L$. Hence $u = 0$ by (**).

(2) This follows from the Bounded inverse theorem.

Theorem 3.14: Let X and Y are Banach spaces and $F: X \to Y$ be a one to one bounded linear map. Then its range $R(F)$ is closed in Y iff $F^{-1}: R(F) \to X$ is bounded.

Proof: Let $Y_1 = R(F)$ be closed in Y. Then $Y_1 = R(F)$ is Banach, since Y is Banach. Moreover $F: X \to Y_1$ is a bijective bounded linear map and X is Banach. Therefore the Bounded inverse theorem gives the continuity of F^{-1}

Theorem 3.15: Suppose X is a Banach space, A and B are closed subspaces of X and $A + B = X$. There exists a constant $\gamma < \infty$ such that every $x \in X$ has a representation $x = a + b$, where $a \in A$, $b \in B$ and $\|a\| + \|b\| \le \gamma \|x\|$.

Proof: Let Y be the vector space of al ordered pairs (a,b) with $a \in A$, $b \in B$ and component wise addition and scaler multiplication, normed by $\|(a,b)\| = \|a\| + \|b\|$

Since A and B are complete, Y is Banach space. The mapping $\Lambda: Y \to X$ defined by $\Lambda(a,b) = a + b$ is continuous, since $\|a + b\| \le \|(a,b)\|$ and maps Y onto X. By the open mapping theorem, there exists $\gamma < \infty$ such that each $x \in X$ is $\Lambda(a,b)$ for some (a,b) with $\|(a,b)\| \le \gamma \|x\|$.

Theorem 3.16: Let X be separable Banach space. Then there exists a closed linear subspace L of the sequence space l_1 such that X is topologically isomorphic to the quotient space l_1 / L

Proof: Let $\{x_n\}$ be a dense sequence in X. Define the mapping $T: l_1 \to X$ by $T(y) = \sum a_n x_n$, $y = \{a_n\}_{n=1}^{\infty}$.

Clearly $\|T(y)\| \le \|y\|$ Let $L = T^{-1}\{0\}$. Define now the mapping τ by $\tau(y+L) = T(y)$. This map is well defined and is continuous, linear, one to one mapping from l_1 / L into X. If T is surjective, then τ is surjective. By applying the open mapping theorem the proof will be complete in this case. Then it is sufficient to prove that T is surjective. Let $x \in X$ and $\|x\| < 1$. Choose x_{n_0} such that $\|x - x_{n_0}\| < \frac{1}{2}$ then choose $n_1 > n_0$ such that $\|2(x - x_{n_0}) - x_{n_1}\| < \frac{1}{2}$.

Arguing inductively, we can finds subsequence $\{x_{n_k}\}$ such that $x = \sum_{k=0}^{\infty} \frac{x_{n_k}}{2^k}$.

Let $y = \{a_j\}$ be defined so that $a_j = \frac{1}{2^i}$ if $j = n_i$

$$= 0 \text{ if } j \ne n_i \text{ all } i$$

Then $y \in l_1$ and $T(y) = x$. This shows that T is surjective.

Now we can mention an application of the open mapping theorem to obtain result about perturbations in different equations.

Example 3.17: Consider the differential equation, $x^{//}(t) + a_1(t)x^{/}(t) + a_2(t)x(t) = y(t)$ ……(*)

Here a_1, a_2 and y are members of $C[a,b]$. An initial value problem (For details see ordinary differential equation by Coddington) for (*) calls for finding a twice continuously differential function x on $[a,b]$ satisfying (*) and satisfying the initial conditions $x(a) = x^{/}(a) = 0$

A standard theorem in differential equations asserts that this initial value problem has a unique solution. We wish to study the dependency of the function x on y and vice-versa.

Let $X = C^2$, the space of twice continuously differential functions. Then X becomes a Banach space under the norm $\|x\| = \max\{\|x\|_\infty, \|x^{/}\|_\infty, \|x^{//}\|_\infty\}$ where $\| \ \|_\infty$ denotes the usual supremum norm in $C[a,b]$. Let $Y = C[a,b]$. Let $T : X \to Y$ be defined by $Tx = y$ where $x^{//}(t) + a_1(t)x^{/}(t) + a_2(t)x(t) = y(t)$. The standard theorem in differential equations that we mention before asserts that T is one to one on X and maps X onto Y. We show that T is a continuous operator.

Let $A = 1 + \|a_1\|_\infty + \|a_2\|_\infty$. Then $\|Tx\|_\infty = \|y\|_\infty \le \|x^{//}\|_\infty + \|a_1\|_\infty \|x^{/}\|_\infty + \|a_2\|_\infty \|x\|_\infty \le A\|x\|$ so T is a continuous. By the open mapping theorem, T^{-1} is also continuous. We can interpret this as saying that small perturbations of the function y will result in small perturbations of the solution $x \in C^2$. This means that such a perturbed solution x_1 will be 'C^2-close' to x, that is, $x_1, x_1^{/}$ and $x_1^{//}$ will be (uniformly) close to $x, x^{/}$ and $x^{//}$ respectively.

4. Conclusion:

Here, the open mapping theorem exhibits the reason why Banach spaces are more satisfactory than incomplete normed spaces. The theorem also gives conditions under which the inverse of a bounded linear operator is bounded. Moreover, Bounded Inverse theorem has shown that just as the inverse of a bijective linear map from a linear space to a linear space is linear and the inverse of a bijective closed map from a metric space to a metric space is closed, the inverse of a bijective, linear and continuous map from a Banach space to a Banach space is linear and continuous. The theorem 4, have shown that open mapping theorem can be used to obtain a theorem showing how to check that two norms on Banach space are equivalent.

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