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INVENTORY MODELS WITH UNCERTAIN DATA

Dr. Rahul Waliv
and
Dr. Hemant Umap

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Dr. Rahul Waliv

Department of Statistics,

Kisan Veer Mahavidyalaya,

Wai, Dist – Satara, M.S., India

and

Dr. Hemant Umap

Department of Statistics,

Yashwantrao Chavan Institute of Science (Autonomous),

Satara, M.S., India

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PREFACE

Basic Optimization Techniques introduces the fundamentals of all the commonly used techniques in Operations Research. Operations Research is the application of the methods of science to complex problems in the directions and managements of large systems of men, machines, materials and money in industry, business, government sector etc. The purpose is to help management in determining its policy and actions scientifically.

The book is designed as a text for an introductory course in Operations Research. Specially target the needs of students who are learning Operations Research as a subject. In various institutes the course may be taught in Statistics, Management Science, Applied Mathematics, Computer Science, Engineering etc. at either the undergraduate level of post graduate level.

The aim is to explain the concepts and simultaneously to develop in readers an understanding of problem solving methods based upon model formulation and solution procedures. Throughout the book, numerous solved business oriented examples have been presented. In writing this book I have benefited immensely by too many books and publications. I express my gratitude to all such authors and publications. I express my sincere gratitude to my guide Dr. H. P. Umap. I wish to acknowledge my thanks to my colloguesfor their help during the preparation of book. At the end let me thank my wife Madhuri for their support andencouragement.

I welcome comments and suggestions from the readers towards the improvement of this book.

\- **Dr. R. H. Waliv**

Content

Chapter 1 Basics of Inventory Control System

1.1 History and Origin of Operations Research (OR):

Operations Research (OR) can be traced all the way back many decades when early efforts to use a scientific approach to technical issues and organizational management have been made.British military resources were very limited during World War II, so it was necessary to effectively allocate these resources to various military operations and activities within each operation.Therefore British military appointed a team of scientists to apply a scientific strategy to tackle tactical and strategic troubles related to land defense and air of the country. Because the crew dealt with military operations research, the work of this crew of scientists in Britain was named Operations Research. The success of operations research in the military has attracted industrial management's interest in finding solutions to their complicated issues of execution type. In this way OR get introduced in industry and business. OR activities encompass transportation systems, libraries, hospitals, urban planning, financial institutions, etc. besides military and business applications. The scientific study of operations of the system is called OR.

After the war, according to needs of various fields many scientists were motivated to pursue research in OR for different fields. Various techniques have been developed in OR. The first technique, recognized as the simplex method, was developed to solve linear programming problems.American mathematician George Dantzing developed this technique in 1947.Since then many techniques are developed. Some of the techniques which are applied to various fields for example linear programming, PERT and CPM, replacement theory**,** game theory,dynamic programming, investment analysis, goal programming, queuing theory, inventory control, etc. These techniques have widen the scope of OR to various fields such as defense, industry, planning, agriculture, public utilities like hospitals, transport system etc.

In India OR came into existence with the opening of OR unit in 1949 at Regional Research Laboratory in Hyderabad. At the same time another OR unit was set up at the Defense Science Laboratory to tackle the problems of stores, purchase and weapon evaluation. An OR unit under Prof. Mahalonobis was established in 1953 in Indian Statistical Institute, Kolkata to apply OR methods in national planning and survey. Towards the application of OR in India, Prof. Mahalonobis made the first important application. He developed the second five - year plan using OR technology to forecast demand trends, resource availability and scheduling complex schemes necessary for developing country's economy.India has been estimated to become self - sufficient in food by reducing food waste by 15 %. To achieve this goal, OR techniques are used. Planning commission used OR for planning the optimum size of the Carville fleet of Indian Airlines.

In short, OR is the collection of modern methods on the problems of men, machines, materials and money systems in the industry, defense,business, etc. According to Churchman *et al.* OR is defined as “The application of scientific methods, techniques and tools for decision making problems (DMP) involving the operations of systems so as to provide these in the control of the operations with optimum solution to the problems”.

1.2 Inventory system in Operations Research:

As mentioned above one of the most applied technique of OR is inventory control. This methodis used to determine maximum profit, economic ordered quantities, reorder levels, stock level etc**.** Many of the Indian companies that use this method are Indian Railways, Hindustan Lever, Delhi Cloth Mills, Defense Organizations, Indian Fertilizer Corporation etc.In large production firms as well as in departmental stores or shops, the storage of items depends on different factors such as demand, time of order, the time lag between orders and actual receipts, deterioration, amelioration, time value of money, inflation, etc. and the impression of these factors. So the problem for the managers and retailers is to have a compromise between overstocking and under stocking. The study of such type of problems is known by the terms “Material management” or “Inventory control”.

The Inventory control may be defined as “the function of directing the movement of goods through the entire manufacturing cycle from the requisitioning of raw materials to the inventory of finishing goods orderly mannered to meet the objectives of maximum customer service with minimum investment and efficient (low-cost) plant operation”. Inventory can be defined as” the stock of goods, commodities or other economic recourses that are stored or reserved in order to ensure the smooth and efficient running of business affairs”. Total investment in inventories in any country amounts to a significant proportion of General Net Production (G.N.P.). This is the situation where inventory control and management is of great concern in any sector of the economy. Some costs are involved in an inventory system, such as holding, set-up, shortage, purchasing, material costs, etc. although inventories are an ideal resource that incurs costs of holding, they are justified by the results of saving in cost of shortage, set-up, and procurement.

The two main issues in controlling an inventory of physical goods are:

When should the physical goods inventory be replenished?

And at the beginning of each time interval, how many physical goods should be purchased or produced?

A problem with the inventory involves making optimal decisions on the questions mentioned above. In other words, an inventory issue addresses a decision that minimizes the total average cost or maximizes the total average profit earned to satisfy the customer's requirements. Inventory is an ideal resource of economic value that needs to be maintained to meet current and future organizational needs. Inventory can be regarded as a necessary non-earning asset that cannot be eliminated. It is called evil because it keeps ties with capital that can be used for alternative purposes and also increases the cost of carrying, but it is a necessary investment to achieve the feasible system of physical goods production, distribution, and marketing.

Inventory can be classified as

i) Physical resources such as raw material, semi-finished goods, spare part, finished goods, lubricants, etc.

ii) Human resources such as unused manpower

iii) Financial resources such as working capital etc.

Inventory holdings are of a different type in the various organizations. For example, if the organization is of the type of manufacturer, it must have raw materials, semi-finished goods, finished goods, etc. If organizations are of the type of service industry such as banks, hospitals, airline firms, etc. A bank must have reserves of cash, tellers, etc. While the hospital must keep no beds, drug stock, etc. Inventory is used to provide customer service in a timely manner and to ensure maximum sales, turnover and customer satisfaction. To maintain inventory a large proportion of capital money has to be invested in it. So balancing the benefit of inventory of resources and the cost of maintaining them is essential, so as to determine an optimal level of inventory of each resource, resulting in minimizing total inventory cost. Inventories such as the army, quantity of food, number of cattle, the quantity of gold and silver, etc. were considered centuries ago to measure a country's wealth and power such an inventory does not require scientific management. Currently, inventory is seen as a potential risk, not as wealth due to faster development and changes in commodity life. Total inventory investment is quite large, so it is of great importance to manufacturing and non-manufacturing organizations. Poor inventory control results in a high cost of production, which can result in enormous losses, so inventory is now considered a measure of business failure. Therefore, a scientific technique known as inventory control is necessary for inventory management. Inventory helps us to control stocks at the lowest level possible, which means that the stock of an existing item is kept at a reasonable level, avoiding adding unnecessary items and removing items that are no longer in use. This helps us to minimize the cost of maintaining inventory.

Three different approaches, check the inventory status,

a) Regular review system, in which order is placed at regular intervals.

b) A fixed order quantity system in which inventory is regularly monitored and when the inventory drops to a particular level, a fixed quantity replenishment order is placed.

c) The optimal replenishment system combines a periodic review system with a minimum order quantity restriction, i.e. where No order will be placed to keep away from putting small orders when the calculated order quantity is less than predicted. Some costs are incurred for each order placed, such as administrative costs, transportation, inspection, etc. If large orders are frequently placed order costs and delivery costs are kept low but stock level and average value are high whereas when small orders are placed, orders and delivery costs are frequently increased but the average stock level is low so we have a tradeoff between these two options to minimize overall total cost however ordered quantity depends on demand pattern, price of items including discounts for larger order, lead time, and various costs, etc.

1.3 Objectives of Inventory control:

i. Maintain optimum stock of goods at a minimum cost, i.e. trade between loss due to the items non-availability and the cost of carrying the commodity.

ii. To obtain an economy in buying, storing, producing and selling.

iii. Maintaining a low level of total inventory investment in accordance with operational requirements.

iv. Inventory should be considered a risky investment, i.e. Investment in some stock may yield higher returns for others and less.

v. To obtain maximum efficiency by minimizing shortage, holding and replacement cost of inventory.

vi. Maintain minimum levels of waste, surplus, scrap, obsolete and inactive items vii) To supply raw material, subassemblies, semi-finished goods, finished goods to their users accordingly to their specifications at the right time and the right price. Improving profitability is an important function of inventory. Inventory helps in maintaining a balance between supply and demand, decoupling and safety stock.

1.4 Merits and Demerits of Inventory:

1.4.1 Merits of inventory:

Inventory is a crucial part of the business. Inventory takes care of requirements till next arrival, probable delay in delivery and sudden increase in demand. The maintenance of the inventory is necessary because it is not possible to supply each item when it is necessary when it is very expensive. Inventory performs an important role for the following reasons:

i. To receive the correct quantity of stock to ensure continuous and smooth production at the exact time of the requirement.
ii. To get benefit from the price discount in ordering large quantities.
iii. To provide satisfactory customer service by supplying most of the requirements without delay.
iv. To carry reserve stocks to avoid stock outs.
v. o take advantage of the economies of transport.
vi. Stabilize production, especially for seasonal commodities.
vii. Planning the overall operational strategy by decoupling successive stages in the chain of acquiring goods, manufacturing items, transportation to the wholesaler and finally serving the customer.
viii. In order to keep up with the changing market, i.e. to provide a hedge against the future value and delivery uncertainties.
ix. To make effective use of capital and storage space.

1.4.2 Demerits of inventory:

i. Holding large stock requires more capital, which leads to a high carrying cost.
ii. Risk of deterioration and spoilage in the longer holding of stocks.
iii. Due to obsolescence, excessive stock can lead to a loss.
iv. Insurance cost and property taxes would be added to the cost of inventories.

1.5 Basic Concepts and Terminology:

1.5.1 Factors affecting inventory:

The main aspects of the inventory are demand and associated costs. The different factors associated with these two factors are discussed below:

i. Number of items:

The number of items held in inventory influenced by limited floor space and budget

ii. Number of stages of inventory:

When parts are stored in more than one stage in a sequential manufacturing process it is called multistage inventory.

iii. Availability of items:

Sometimes supply may be affected badly due to the market conditions which in turn affect inventory position in an enterprise.

iv. Government norms:

The government has established certain policy standards for specific items such as imported items, explosive items, highly inflammable items, and essential items.

1.5.2 Various inventory parameters used in inventory control:

I. Various costs involved in inventory control system:

i. Purchase cost or production cost:

This is the cost which paid to the supplier for the commodity to be purchased or production cost if the manufacturer. This price fluctuates so the average cost is considered for planning purposes. This cost plays an important role when purchasing item has a discount for a large number of items

ii. Procurement cost:

These costs incurred when order is placed known as ordering cost or related to the initial preparation of the production system known as set up cost. This cost is corresponding to the orders placed. This cost includes salaries of the employee, tendering, paperwork, postage, transportation cost, inspection cost, processing payments, etc. This cost is per set up or per order.

iii) Holding costs or carrying costs:

The cost of holding items is related to this cost. This cost is commensurate with the number of items stored. In general, this cost ranges from 15-30 percent of the price of the commodity concerned and includes:

a) Storage and handling charges

b) Interest or capital investment

c) Insurance expenses

d) Loss due to depreciation, spoilage, pilferage, deterioration, obsolescence

e) Warehouse rent

f) Cost of safety measures (This cost is represented as per unit of item held per unit of time).

iv. Shortage cost:

These are penalty costs incurred due to stock out. This can be on selling side (backorder cost or lost sale cost) and for manufacturer (backlog cost or non-backlogging cost).backorder cost or backlogging cost arises when customer wait till get supplies i.e. unfulfilled demand can be fulfilled later. Lost sale cost or no backlogging cost incurred when customers don't wait for supply and go to another option i.e. demand is not fulfilled. This cost includes the cost of production stoppage, overtime payment, expenditure on special order with a higher price, idle machine, loss of goodwill, loss of profitability, etc.

v. Selling price:

This cost includes revenue from the selling of commodities. This price may be a fix or variable depends on a quantity discount.

vi. Cost of Operating the information processing system:

Records of changes in stock levels can be updated either by hand or by computer. In case, when the inventory levels are not recorded daily, the opening cost is incurred to obtain in physical counts of inventories. These operating costs are usually fixed over a wide volume range of product.

II. Demand:

Demand refers to the amount of a commodity needed at a given time. Generally, it cannot be controlled directly and in many cases even indirectly also. It usually depends on the decision of people outside the organization. Demand can be categorized according to its size and pattern. Demand size refers to both the demand magnitude and the quantity dimension. It partly depends upon the stock level (initial or on hand), time, the selling price of the item and advertisement, etc. Demand can be deterministic or probabilistic

a) Deterministic demand: in this case, the quantity required is known with certainty over a subsequent period of time and can be expressed in terms of cost over equal periods of time.

b) Probabilistic demand: This happens when the required quantities are not known for certain periods of time, but the pattern can be expressed through probabilistic distribution.

III. Lead time:

Time between requisition for an item and its receipt. It can be deterministic or probabilistic. It includes four components

a) Administrative lead time

b) Supplier lead time

c) Transportation lead time

d) Inspection lead time

IV. Time horizon:

This refers to the planning period during which inventory should be checked. It may be a period of finite or infinite. Inventory planning is usually carried out on an annual basis.

V. Echelon:

These are number of supply points.

VI. Constraints:

Constraints are the limitations imposed on the inventory system. Constraints may be imposed on the amount of investment, available space, the amount of inventory held, average inventory expenditure, number of orders, etc. These constraints may be fuzzy in nature i.e. data for constraints goals may be imprecise and vague. There may be also some chance constraints in the inventory system, i.e. the minimum probability of satisfying a constraint is specified or some parameters in the constraint(s) may be probabilistic.

VII. Deterioration:

It is defined as "decay, evaporation, obsolescence, and loss of unity or marginal value of a commodity that results in the decreasing usefulness from the original condition". Vegetables, food grains, gasoline and semiconductor chips, etc. are examples of such commodities.

VIII. Damageability:

It is defined as "The damage when the items are broken or lose their utility due to the accumulated stress, bad handling, etc."The quantity of damage triggered by stress varies relying on the stock size and the stress period. Examples of such commodities are items made from glass, china-clay, ceramic, mud, etc.

IX. Perishable items:

These are those, which have a finite lifetime (fixed or random). Fixed lifetime commodities like human blood and so on have a deterministic shelf-life while a random lifetime scenario assumes that a random variable is the useful life of each unit. The random lifetime scenario is closely linked to an inventory that experiences continuous physical deterioration or decline.

X. Salvage:

Some items were partially spoiled or damaged during the shortage, i.e. some items lost their usefulness. In a developing country, however, it is usually observed that some of these are sold to a section of customers at a lower price (less than the purchase price), and this gives the management some revenue. This income is called the salvage value.

XI. Fully back- logged / partially back- logged shortages:

Sales or goodwill might also be lost through a delay or complete refusal to meet the demand in the course of the stock-out period. If the unfulfilled demand for the goods can be fully satisfied at a later date, then there will be a fully backlogged shortages i.e. it is assumed that no customer back away during this period and the demand of all these waiting customers is met at the beginning of the next period. Again, it is generally found that some of the customers wait for the product and others return away during the stock-out period. When this happens, the phenomenon is called partially backlogged shortages.

Among the above factors, major factors influence inventory are demand, economic parameters and deterioration.

1.5.3 Types of inventory:

i. Lot size inventory:

The inventory must meet the average demand during successive replenishment. The quantity of such stock depends on the size of the lot of production, the economic quantity of shipments, limited storage space, lead time, cost carrying inventory, discounts on price quantities, etc.

ii. Pipeline inventory:

Inventory movement cannot be instantaneous, inventory rises as inventory items are shipped to remote centers and customers, from production centers such inventory is called as process inventory. It consists of actually working material and moving between the workplace centers. This inventory must be maintained without delay because the quantity of such inventory depends on the time required for shipments and the nature of demands when the supply is in transit.

iii. Buffer inventory:

This type of inventory is designed with some probability distribution to protect against demand and lead time uncertainties. This inventory helps reduce unpredictable shortages that can result in a high cost of penalties. To avoid losses due to future uncertainties, this inventory is maintained as additional stock in regular stock. The level of such inventories is determined by the desired trade-off between demand and supply uncertainties protection and the level of security stock investment.This inventory provides protection against the following uncertainties: a) sales during the replenishment period exceeding the forecast. b) With regard to replenishment delay.

iv. Seasonal inventory:

This inventory is required for commodities on the market with a seasonal demand pattern and the production of which is not uniform (i.e. varies over time) such as fashion items, agricultural commodities, children's toys, etc. In these cases, the manufacturer will be dealt with maximum demand if the manufacturing facility is unable to meet the demand for a period of time. During the period of low demand, these inventories stored to meet peak demand. The quantity of such inventory is determined by balancing the holding and shortage cost of seasonal inventory.

v. Decoupling Inventory:

This inventory includes separating inventory within a manufacturing process so that the inventory associated with one stage of a manufacturing process does not slow down other parts of the process.This inventory deal with issues such as equipment breakdowns or uneven machine production rates affecting output because one part of the production line works at a different speed than another.

This inventory serves the following purposes:

a) Inventory is required to reduce dependency among successive operation stages

b) One organization unit can schedule its operation independent of another for example steel making organization, steel melts production can be scheduled separately from rolling and forging.

1.5.4 Inventory Classification Based on Service Utility:

1.5.4.1 Manufacturing Inventory:

This refers to the inventory held by manufacturing firms this can be subdivided as

a) Production inventory: these are items which go to final commodity such as raw material, components, and sub-assemblies purchase from outside.

b) Work in progress inventory: This includes all items at various stages in semi-finished form.

c) Finished goods inventory: This included commodity for shipment to the user.

d) Maintenance, Repair or operating inventory: Supplies consumed in the system of manufacturing, however not part of the end commodity or central to the output of the company.This inventory consists of consumables (e.g. cleaning, laboratory or office supplies), industrial equipment (e.g. compressors, pumps, valves) and components for plant protection (e.g. gaskets, lubricants and repair equipment) as well as computers, fittings, furniture, etc.**)**

e) Miscellaneous Inventory: these are items not covered by the category mentioned above. This includes the outdated and unsalable commodity that arose from the office's main production, stationery, and other items.

1.5.4.2 Non-Manufacturing Inventory:

This inventory is necessary to provide an organization with economical and efficient operations. These are four types

a) Lot size: this refers to a lot size purchase. This is used for

i) Obtaining quantity discount

ii) Reducing transport and receiving costs for example buying the bulk of oilseeds during the oilseed season would be cost-effective for the oil mill.

b) Anticipation stock: This is required to meet predictable changes in demand or in the availability of raw material, for example Purchase of wheat in a wheat season for sales of seed preservation commodities throughout the year.

c) Fluctuation stock: This is required to ensure ready supplies to the customer when their fluctuation in their demand.

d) Risk stock: These are needed to prevent the risk of breakdown of production.

1.5.5 Classification of inventory depending on its use:

1.5.5.1 Direct Inventory:

Objects that play a direct role in the manufacturing and turn out to be an integral part of finished goods are called direct inventories.These were classified as follows:

a) Raw material: these are basic materials that have not been converted, but kept in stock for manufacturing use. Examples are steel, copper, lead, tin, rubber, etc. these are required for the following purposes

i) For economical bulk purchasing

ii) To make it possible to change rates of production

iii) To serve as buffer stock against delay in transportation

iv) To cope with seasonal variations in supplies

b) Work in process inventories: These are materials in the partially completed condition of manufacture. At the end of the first operation, raw materials become work in process and remain classification until they become the final commodity. Usually, the materials on conveyors, trucks, etc. are considered work in progress. These are provided for the following purposes

i) Providing an economic lot of production

ii) To enable a variety of commodities

iii) To replace the waste

iv) Keeping production uniform even during sales variations

c) Finished goods inventories: These are final commodities ready for delivery to customers. Commodities usually leave work in progress, classification and enter the finished goods category at the end of the final inspection.

1.5.5.2 Indirect inventories:

These include items essentially required for manufacturing, but not becoming an integral part of finished goods. These are grouped as:

a) Tools: These consist of

i) Standard tools to be used on machines such as saws, drills, taps, milling cutters, etc.

ii) Hand tools such as hammers, needles, spanners, etc.

b) Supplies: These include materials required for the running of the plant or company commodities. Supplies may include:

i) Welding, Soldering, and tinning material (such as electrodes, gas, welding rods, flux, etc.)

ii) Shipping containers (such as bags, glass bottles, cardboard boxes, drum, etc.)

iii) Oils and greases

iv) Electric supplies

v) Printed forms

c) Machinery spares: These are the material required in maintenance of machines, for example bearing, belts, oil seals, etc.

1.6 Inventory modeling:

The intention of this research is to develop the operating inventory models that should be used with mathematical analysis to control the inventory system. Therefore, the task is to design a mathematical model of the real-life inventory system in various environments. However, such a mathematical model is based on various assumptions and approximations.

The methodology for inventory modeling involves the following steps:

a) The inventory situation is examined carefully and characteristics and assuming relating to a situation are listed.

b) The total relevant cost equation is developed in narrative form as follows:

Total annual cost=cost of item+ procurement cost+ stock carrying cost+ stock out cost

c) Finally cost equation is optimized by finding optimum values for quantity to be replenished, reorder point and the total relevant cost.

1.7 Environments for developing inventory models:

The parameters of the inventory, the objective goals are assumed to be deterministic and fixed in most existing inventory models. But in reality they are ambiguous, either random or imprecise. To tackle randomness stochastic environment is used. To handle impreciseness fuzzy environment is used. So In general, inventory control parameters are considered in the development of an inventory model in three environments such as deterministic, stochastic, fuzzy and fuzzy- stochastic environments.

a) Deterministic (crisp) Environment:

Parameters of the inventory are considered known and constant in the deterministic environment. There is so much literature available in research journals and books on the deterministic inventory model.

b) Fuzzy Environment:

In some situations parameters like demand, holding cost, set-up cost, purchasing price, storage area, production rate, etc. are vague and imprecise. Their values may vary within the ranges such cases are tackled by using a fuzzy environment.

1.8 Fuzzy Preliminaries:

Definition: If X is a collection of objects denoted by x, then fuzzy set $\tilde{B}$ in X is a set of ordered pairs:

$\tilde{B} = \{(x, \mu_{\tilde{B}}(x)) \mid x \in X\}$

$\mu_{\tilde{B}}(x)$ is called the membership function or grade of membership of x in $\tilde{B}$ that maps X to the membership space M. The membership function range is a subset of the non-negative real numbers with a finite supremum.

The support of a fuzzy set $\tilde{B}$, denoted by S($\tilde{B}$) given by

$S(\tilde{B}) = \{x \mid \mu_{\tilde{B}}(x) > 0\}$

- **α-Level Set:**

α-Level Set denoted by B_α is given by $B_\alpha = \{x \in X \mid \mu_{\tilde{B}}(x) \geq \alpha\}$ and it is called 'strong α -Level Set' if $B_\alpha = \{x \in X \mid \mu_{\tilde{B}}(x) > \alpha\}$

- **Complement:**

The membership function of the complement of a normalized fuzzy set $\tilde{B}$, $\mu_{\tilde{B}^c}(x)$ is defined by

$\mu_{\tilde{B}^c}(x) = 1 - \mu_{\tilde{B}}(x), x \in X$

- **Cardinality:**

For a finite fuzzy set $\tilde{B}$, the cardinality $|\tilde{\boldsymbol{B}}|$ is defined as

$|\tilde{B}| = \sum_{x \in X} \mu_{\tilde{B}}(x)$

- **Convexity:**

Let $\tilde{B}$ be a fuzzy set in X. Then it is convex if and only if for any $x_1, x_2 \in X$,

the membership function of $\tilde{B}$ satisfies the inequality

$$\mu_{\tilde{B}}(\lambda x_1 + (1 - \lambda)x_2) \geq \text{Min}(\mu_{\tilde{B}}(x_1), \mu_{\tilde{B}}(x_2)) \quad \text{for } 0 \leq \lambda \leq 1$$

- **Union:**

The membership function $\mu_{\tilde{D}}(x)$ of the union $\tilde{D} = \tilde{B} \cup \tilde{C}$ is point wise defined by

$\mu_{\tilde{D}}(x) = \max\{\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x)\}, x \in X$

- **Intersection:**

The membership function $\mu_{\tilde{C}}(x)$ of the union $\tilde{E} = \tilde{B} \cap \tilde{C}$ is point wise defined by

$$\mu_{\tilde{E}}(x) = \min\left\{\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x)\right\}, x \in X$$

In fuzzy set theory, there are different membership functions defined. Some of the membership functions like linear membership function, hyperbolic inverse membership function, exponential membership function and n^{th} parabolic membership function are used to develop fuzzy inventory models.

- **Fuzzy number and Membership functions:**

- **Fuzzy number:**

A fuzzy number is a special case of a fuzzy set. Different concepts and properties of fuzzy numbers are encountered in the literature but they all agree on that a fuzzy number represents the conception of "a set of real numbers close to a "where B is the number being fuzzified.A fuzzy number is a fuzzy set in the universe of discourse X that is both convex and normal.

Membership function:

The fuzziness is well described by its membership function. Membership function represents the degree of reality in fuzzy set theory. Various membership functions are available to represent fuzziness of parameter in fuzzy set theory. Some of them are linear, hyperbolic inverse, exponential and n^{th} parabolic membership functions which are used to develop inventory models. Types of membership function

- **Linear Membership function:**

$$\mu_{ci} = \begin{cases} 1 & ; u > ci \\ 1 - \left(\dfrac{ci - u}{P_{ci}}\right) & ; ci - P_{ci} \le u \le ci \\ 0 & ; u > ci - P_{ci} \end{cases}$$

$$\mu_{ci}^{-1}(\alpha) = ci - (1 - \alpha)P_{ci}$$

- **Hyperbolic inverse Membership function**

For each fuzzy parameter $\tilde{A}$ the corresponding hyperbolic membership function is defined by

$$\mu_A(u) = a\tanh^{-1}((A(u) - b)c) + \frac{1}{2}$$

The hyperbolic inverse membership function can be determined by asking the decision maker to specify the three points A(0), A(0.25) and A(0.5) within A_{max} and A_{min}.

$$\mu_{ci}^{-1}(\alpha) = b_{ci} + \frac{1}{C_{ci}}\tanh\left[\frac{\alpha - \frac{1}{2}}{a_{ci}}\right]$$

- **Exponential Membership function**

$$\mu_{ci}(u) = \begin{cases} 1 & ; u > ci \\ \left(\dfrac{q^{t(ci-u)/P_{ci}} - q^t}{(1 - q^t)}\right) & ; ci - P_{ci} \le u \le ci \\ 0 & ; u > ci - P_{ci} \end{cases}$$

where 0<q<1, t>0, i =1,2.Let q=0.5 and t=1 then

$$\mu_{ci}^{-1}(\alpha) = ci - P_{ci}\log_{0.5}(0.5 + 0.5\alpha)$$

- **n^{th} Parabolic Membership function**

$$\mu_{ci} = \begin{cases} 1 & ; u > ci \\ 1 - \left(\dfrac{ci - u}{P_{ci}}\right)^n & ; ci - Pci \le u \le ci \\ 0 & ; u > ci - P_{ci} \end{cases}$$

$$\mu_{ci}^{-1}(\alpha) = ci - (1 - \alpha)^{\frac{1}{n}} P_{ci}$$

1.9 Methodologies:

1.9.1 Theory of Fuzzy Non Linear Programming (FNLP):

a) FNLP with fuzzy objective and fuzzy constraints

Consider a model of the problem “maximize an function subject to constraints,” namely the “linear programming model”

$$\max z = d^T x$$
$$\text{such that } Bx \le c \text{ - - - -}(1)$$

Where d and x are n-vectors, c is an m-vector and B is an m x nmatrix.

If objectives as well as constraints in above model are imprecise and vague then the Bellman and Zadeh approach is used. Fuzzy sets represent imprecise and vague goals as well as constraints then membership function of decision is then defined as follows

Definition: Let $\mu_{\tilde{c}_i}(x), i = 1, 2, \ldots, m, x \in X$ be membership functions of constraints defining the decision space and $\mu_{\widetilde{d_j}}(x), j = 1,2, \ldots . n, x \in X$.The membership functions of objective goals. A decisionis then defined by its membership function$\mu_{\tilde{E}}(x) = \mu_{\widetilde{d_j}}(x) * \mu_{\widetilde{c_j}}(x), i = 1,2, \ldots, 3, j = 1,2, \ldots, n$ where * denotes an appropriate, possibly context dependent, “aggregator” (connective).

Let M be the set of points $x \in X$ for which $\mu_{\tilde{E}}(x)$ attains its maximum, if it exists. Then Mis called the maximizing decision**.** If $\mu_{\tilde{E}}(x)$ has a unique maximum at x_M, then the maximizing decision is a unique crisp decision that can be interpreted as an action belonging to all fuzzy sets representing either constraints or objectives with the highest possible degree of membership. (That could be quite low)

Assume a situation as described in Definition 1, i.e. The decision maker's goal and constraints can be expressed as a fuzzy set. In such a situation a model (1) becomes

$$d^T x \tilde{\leq} z$$

$$Bx \tilde{\leq} c \text{-----}(2)$$

Here $\tilde{\leq}$ denotes the fuzzified version of $\leq$ and has "essentially less than or equal" linguistic interpretation. The objective function in (1) might have to be written as a minimizing goal in order to consider z as an upper bound. by following substitution model (2) becomes

$$\begin{pmatrix} d \\ B \end{pmatrix} = H \text{ and} \begin{pmatrix} z \\ c \end{pmatrix} = I$$

Find X such that

$$Hx \tilde{\leq} I \text{---}(3)$$

Each of the m + 1 rows of (3) will now be represented by a fuzzy set, the membership functions of which are$\mu_i(x)$ which can be interpreted as the degree to which x satisfies inequality $Hx \tilde{\leq} I$. Using Definition, the membership function of the fuzzy set “decision” of the model (3) is

$$\mu_{\tilde{D}}(x) = \min_{i=1}^{m+1} \mu_i(x) \text{----}(4)$$

Assuming the decision maker is not interested in a fuzzy set, but in a crisp set "optimal" solution x_0, we could propose a "maximizing solution" to (4)which is the solution to the possibly nonlinear programming problem

$$\mu_M(x_0) = \max_{x\geq 0} \min_{i=1}^{m+1} \mu_i(x)$$

Now we have to specify the membership functions $\mu_i(x)$.

Using the simplest linear membership function increasing over the tolerance interval (d_i, $d_i + p_i$)

$$\max_{x\geq 0} \min_{i=1}^{m+1} \left(1 - \frac{(Hx)_i - d_i}{p_i}\right)$$

Where p_i, aresubjectively chosen constants of admissible violations of the constraints and the objective function.

Introducing one new variable α which corresponds essentially to $\mu_{\tilde{D}}(x)$ in (4), then

$$\max \alpha$$
$$\text{s.t.} \quad (\alpha - 1)p_i + (Hx)_i \leq d_i, i = 1, 2, ..., m ----(5)$$
$$0 \leq \alpha \leq 1$$

If the optimal solution to (5) exists, then (4) and (3) will have maximizing solution, assuming linear membership function.In general, it is possible to use any nonlinear membership function instead of linear membership function.

b) FNLP with fuzzy objective, constraints and coefficients

A crisp non-linear programming problem may be defined as follows:

$$\text{Minimize } m_0(x, d_0)$$
$$\text{subject to}$$
$$m_i(x, d_i) \leq c_i \qquad i = 1, 2, 3, ---, m$$
$$x \geq 0$$

Where $X=(X_1,X_2,\ldots,X_n)^T$ is a variable vector. m_o and m_i are algebraic expressions in x with coefficients d_o and d_i respectively. After introducing fuzziness in crisp parameters, the problem in a fuzzy environment becomes

$\tilde{M}$inimize $m_0(x, \tilde{d}_0)$

subject to

$m_i(x, \tilde{d}_i) \le \tilde{c}_i \qquad i = 1, 2, 3, ---, m$

$x \ge 0$

In fuzzy set theory the objective, coefficients and constraints are defined by their membership functions which may be linear or non-linear. According to Bellman and Zadeh (1970) and following Carlsson and Krohonen (1986), problem is transformed to

Maximize α

subject to

$g_0(x, \mu^{-1}_{d_0}(\alpha))$

$g_i(x, \mu^{-1}_{d_i}(\alpha)) = \mu^{-1}_i(\alpha) \qquad i = 1, 2, 3, ---, m$

$x \ge 0$

where membership functions of fuzzy coefficients are $\mu_{d_i}(x)$ and of fuzzy objective$\mu_{d_0}(x)$ and fuzzy constraints are $\mu_i(x)$, (i=1,2,3,----m). Here 0<α<1 is an additional variable which is known as aspiration level.

1.9.2 Theory of Intuitionistic fuzzy optimization:

When the 'β' degree of rejection (non-membership) is defined simultaneously with 'α' the degree of acceptance (membership) and when both these degrees are not complementary to each other than IF sets can be used as a more general tool for describing uncertainty. It is possible to represent objectives and constraints by IF set i.e. by pairs of membership $\mu_i(x)$ and rejection $v_i(x)$functions. Each of the m + 1 rows of (3) will now be represented by a intuitionistic fuzzy set then the membership function of the fuzzy set "decision" of the model (3) is

$$\mu_{\tilde{D}}(x) = \min_{i=1}^{m+1} \mu_i(x), \mu_{\tilde{D}}(x) \le \mu_i(x)$$

$$v_{\tilde{D}}(x) = \max_{i=1}^{m+1} v_i(x), v_{\tilde{D}}(x) \ge v_i(x)$$

Different from Fuzzy Linear programming since Conjunction of intuitionistic fuzzy sets is defined as intersection of membership function and union of non-membership function.

Assuming that the decision maker is interested not in a fuzzy set but in a crisp "optimal" solution x_0, we could suggest to him the "maximizing solution" to (4), which is the solution to the possibly nonlinear programming problem i.e. an IFO problem is formulated as to maximize the degree of acceptance of IF objective(s) and constraints and to minimize the degree of" rejection IF objectives and constraints.

$$\max_{x \geq 0} \min_{i=1}^{m+1} \mu_i(x)$$

$$\min_{x \leq 0} \max_{i=1}^{m+1} v_i(x)$$

Subject to $v_i(x) \geq 0$,

$\mu_i(x) \geq v_i(x)$

$\mu_i(x) + v_i(x) \leq 1$

Solving above problem is equivalent to solve

$\max = \alpha - \beta$

s.t. $\mu_i(x) \geq \alpha$

$v_i(x) \leq \beta$

$\alpha \geq \beta, \beta \geq 0, \alpha + \beta \leq 1$

If the optimal solution to above exists then it is the maximizing solution to the model (3), assuming intuitionistic fuzzy set

1.10 Review of Literature for Crisp Inventory models:

Inventory models developed in crisp environment are called as crisp inventory models. While developing such models demand and deterioration are key factors.

a) Literature Review for crisp according to Demand

A sufficient number of crisp inventory models are developed using different types of demands in literature. So many researchers have used different types of demands such as constant, price dependent, stock dependent, time dependent, exponential, ramp type etc. Generally, demand is assumed to be constant in traditional inventory models, but it rarely occurs in real life situations.Bhunia and Maiti (1997) built an inventory model with linear time-dependent demand in this regard.Inventory models with time-dependent demand and weibull distribution deterioration have been built by Wu (2002) and Sharma (2013). Pervin *et al.* (2018) provided optimal retailer's replenishment choices to decaying items along with time-dependent demand.Seema Sharma (2018) and NabenduSen (2018) also considered time dependent demand for construction of inventory models.

Lee *et al.* (2002), Sharma *et al.* (2012) and Krishnaraj (2012) developed inventory models with deterioration rate expressed using weibull distribution and power patterns demand.

With weibull distribution deterioration and quadratic demand, Ghosh *et al.* (2004) and Begum *et al.* (2010) built inventory model.

Chen and Ouyang (1999) have constructed EOQ Model with time dependent deterioration rate andramp type demand. Deng (2005), Jain *et al.* (2007) developed inventory model based on ramp type demand and expressed deterioration rate using three-parameter weibull distribution. Choudhury *et al.* (2015) proposed a ramp-type demand inventory model for decaying items. Poonam Mishra (2017) proposed an integrated inventory model with ramp type demand and capacity utilization dependent holding costs. Huang and Wang (2011) built an EOQ Model with verhulst's demand.

Krishnaraj (2013) developed an inventory model with stock dependent demand and by usingweibull distribution for expressing deterioration. Choudhury *et al.* (2015) designed an inventory model with stock dependent demand for decaying items for two components and holding costs changes according to time. Mishra (2017) formulated stock dependent demand inventory model.

Begum *et al.* (2010) built an EOQ model with weibull distribution deterioration and price-dependent demand.Xiaoyan *et al.* (2015) developed a price - dependent inventory model to optimize the quantity of liquidation and promotional unit price.Alfares*et al.* (2016) derived an inventory model based on price dependent demand, time dependent holding cost, and order size dependent purchase cost.Rastogi (2017) also used selling price dependent demand to develop inventory model for manufacturing process. Researchers like Sana(2008), Palanivel (2015), Geetha (2016)studied effect of price and advertisement dependent demandon inventory model.

Chan *et al.* (1999) and Lee (2009) constructed inventory model for general time dependent demand in there paper they used exponentially increasing demand in example for illustration purpose. Exponential increasing demand is used by Mehta *et al.* (2003). They have formulated a lot-size inventory model with exponentially increasing demand by allowing complete backlogging for decaying items. Sarkar (2013) developed inventory model for manufacturing process with exponentially increasing demand. Also,Yadav*et al.* (2013) and Shukla *et al.* (2010) used similar demand.

b) Literature Review for crisp according to Deterioration Rate

Another important parameter for inventory management is deterioration. Initially, in classical inventory models it was assumed that shelf life of product is infinite and if deterioration rate low and slow then it was ignored. Deterioration is another important parameter of inventory management. Initially, in classical inventory models it was assumed that shelf life of product is infinite and if deterioration rate low and slow then it was ignored. It is assumed that the rate of deterioration is constant after some time, but researchers came to know that it affects inventory and plays important role for determining economic ordered quantity so various types of deterioration rates have been considered to construct inventory models.

An EOQ model for an inventory with exponential decay has been developed by Ghare and Schrader (1963), Misra (1975), Shah (1977) and Tadikamalla (1978) etc. It was later observed that the rate of deterioration changes with time, such cases addressed by time dependent deterioration rate. Some of the researchers such as have made use of weibull distribution for expressing time dependent deterioration rate. Covert and Philip (1973), Elsayed and Teresi (1983), Chakravarty and Balakrishnan(1997), Chakrabarty *et al.* (1998),Wu J. W. *et al.* (1999,2000),Chung *et al.* (2000),Wu K.S.(2001), Mondal *et al.* (2003) and Wang (2004) have made use of weibull distribution while Rao (2016) used a generalized Pareto distribution to model time dependent deterioration rate. For more details regarding crisp inventory model one can refer books of Churchman *et al.* (1957), Arrow *et al.*(1958),Hadley and Whitin(1963) and Naddor (1966).

1.11 Review of Literature for Fuzzy Inventory models:

All the above models developed in crisp environment. But in real life situation crisp environment seen rarely, in general fuzzy environment occurs frequently. In this regard, Roy and Maiti (1995) developed inventory model in which storage space, cost coefficients and budgetary cost are represented by fuzzy numbers. Ishii and Konno (1998) developed fuzzy inventory models by introducing fuzziness in shortage cost. Roy and Maiti (1998) derived multi item inventory model for decaying items in which warehouse space, inventory costs and objectives of maximizing profit and minimizing waste cost are assumed to be vague and imprecise. Chiang Kao and Hsu (2002) developed inventory model with fuzzy demand. Mandal *et al.* (2005) developed model with cost parameter, objective function and constraints considered in fuzzy environment. Roy *et al.* (2008) considered inventory costs, storage area, and budget allocation to be fuzzy. Prasath and Seshaiah (2011) taken unit cost is in fuzzy environment and demand is function of unit cost.Kumar *et al.* (2015) obtained solution of fuzzy inventory model by signed distance method and centroid method. Hardik N. Soni (2015) built the inventory model that included fuzzy random demand, variable lead time with lost sales. Alok Kumar (2016) obtained EOQ model with fuzzy logic for new commodity and demand rate obey innovation diffusion process. Waliv and Umap (2018) used ramp type demand for construction of two-warehouse fuzzy inventory model.

1.12 Review of Literature for Stochastic Inventory models:

Inventory models built in stochastic environments are called as stochastic inventory models. The first stochastic model developed is now known as ‘News Boy Problem ‘obtained during World War II. Barankin and Denny (1960), Brown (1964) and Pierskalla (1969) studied Inventory models for perishable or decaying items subject to exogenous demands with stochastic horizon. Browne and Zipkin (1991) built inventory model, where demand rate at each instant is determined by an underlying stochastic process. Gallego *et al.* (1994) constructed inventory model with demand having compound Poisson distribution. Berman *et al.* (1999) developed stochastic inventory model. Miranda *et al.* (2004) derived EOQ model with stochastic demand. Ouyang *et al.* (2004) considered two models. While developing the first model it is considered that the distribution of lead time follows a normal distribution. The second model assumes itis distribution free. Recently Leiva *et al.* (2016) proposed inventory model with demand having Birnbaum–Saunders (BS) distribution. Chen *et al.* (2016) constructed dynamic stochastic inventory model. Gharaei *et al.* (2017) provided an inventory model with stochastic constraints for imperfect quality products. Mehdi *et al.* (2018) constructed multi sourcing inventory model with stochastic demand. Fawzat *et al.* (2018) developed dual warehouse channel warehouse inventory model with stochastic demand.

1.13 Review of Literature for Fuzzy-Stochastic Inventory models:

To get closer to real life situation some researchers construct fuzzy stochastic inventory models. In this environment, very few models have been developed. Das *et al.* (2004) formulated fuzzy stochastic inventory model by considering demand and budget resources as random and available storage space, total expenditure as fuzzy. Panda *et al.* (2005) developed EOQ model in fuzzy random environment by considering unit purchasing cost, inventory cost and investments as random variable with normal distribution and total cost and constraints goals for storage area as fuzzy in nature Waliv and Umap (2017) developed a multi item profit maximization fuzzy stochastic inventory model.

Chapter 2 Two Warehouse Fuzzy Inventory Model for Deteriorating Items

2.1 Introduction:

Traditional inventory models for decaying items are made mainly for single storage facility with unlimited capacity. In previous chapter inventory models are constructed only for single warehouse system but in some situations two of more warehouses needed to hold inventory.In practice, retailers have to buy a large number of products owing to the following circumstances.

i. An attractive price discount in bulk purchasing and to avoid high inflation rates
ii. The costs of processing goods are greater than the other costs associated with the inventory
iii. There is a very huge demand for items
iv. Frequent procurement has few issues
v. Maintaining a higher inventory of multiple products helps to make ample profit
vi. The customer's goodwill is required to survive in an increasing environment of business competition.

Due to limited capacity of the existing storage facility, large volumes of items cannot be accommodated. So additional storage requires with abundant space to hold a massive stock and excess items. Such additional storage is known as Rented Warehouse (RW) and existing owned warehouse is known as Own Warehouse (OW). Therefore study of two storage facility inventory models is important.

So to minimize this extra cost, customer demand is satisfied by using items in RW and after RW get empty, items in OW are used. This type of model was introduced by Hartely (1976). Sarma (1983) developed an infinite rate inventory model for manufacturing process without shortages. Dave (1988) discussed finite and infinite replenishment rates, inventory models, correcting the Sarma's (1983) model errors and provided a complete solution. Further, Goswami and Chaudhuri (1992) regarded the inventory models with or without shortages for linearly rising time-dependent demand. Correcting and modifying the assumptions of Goswami and Chaudhuri (1992), Bhunia and Maiti (1994) analyzed and graphically demonstrated the same inventory model with sensitivity analysis for changes in demand parameters on the best possible average cost and cycle duration.Maiti *et al.* (2007) investigated the performance of two storage facility for supply chain. Saha *et al.* (2012) formulated a two-storage facility model based on the idea that the sale takes place from OW and that the items sold are continuously substituted by items stored in RW. Rented Storage facility located on the marketplace is named as OW and another Rented Storage facility located a short distance away from the marketplace is named as RW. Niraj Kumar (2013) developed a K-release rule inventory model for two storage facilities. All above mentioned models discussed the cases of non - decaying items.

Some items perceive the natural phenomenon of decay.The researchers therefore, regarded the impact of decay on the item during the development of the inventory model. Decaying is defined as damage, spoilage, dryness, vaporization, etc. It leads to a reduction in the utility of the item. The rate of decay is commensurate with time and temperature.For different items, it may be different. Sarma (1987) presented a model with an infinite rate of replenishment and shortages, the decaying in both storage facilities. Also, Pakkala and Achary (1992) developed the two storage facility inventory models for decaying items with finite replenishment and shortages. In continuation, throughout the past few decades, a number of interesting research articles have been published for two warehouse inventory model with decaying items. One could refer to works of Bhunia and Maiti (1998), Yang (2004, 2006, 2012), Zhou and Yang (2005), Lee (2006), Hsieh *et al.* (2008), Niu and Xie (2008), Ouyang *et al.* (2008),Chung *et al.* (2009), Lee and Hsu (2009), Jaggi and Verma (2010), Liang and Zhou (2011), Saha *et al.* (2012), Zhong and Zhou (2013) for a thorough study. Valliathal and Uthayakumar (2013) proposed two-storage facility inventory model for decaying items with infinite time horizon and shortages. Rao *et al.* (2015) analyzed an EOQ model by considering effect of inflation and permissible delay in payments for decaying items.

In classical inventory models, it was considered that parameters such as demand, holding cost, shortage cost, total floor space and total budget allocation for replenishment are known and constant. Inventory models developed using these assumptions are known as crisp or deterministic inventory models. Different algorithms were developed to solve inventory problems. Inventory parameters in the real-life scenario may be uncertain in non-stochastic sense.For example in the competitive market; the business with predefined fixed budget cannot be carried out always. A decision-maker (DM) may initially start with some fixed quantity, but DM is compelled to raise some more assets in order to satisfy the sudden rise in demand at a later point or to take benefit of the sudden drop in commodity prices.Therefore, in this case the budget allocation is imprecise similar may be the case to the storage area. DM often has vague goals such as ‘‘profit should be larger than or equal to a certain value.’’ In such situations it is better to develop inventory models in fuzzy environment rather than crisp environment.To deal quantitatively with imprecise information in the decision-making process, Bellman and Zadeh (1970) first introduced the notation of fuzziness.Tanaka *et al.* (1974) applied fuzzy ideas to the issue of decision-making taking the goals as fuzzy goals over the α-cuts of a fuzzy constraint. Zimmermann (1978) introduced classic algorithms to solve the problem of fuzzy linear programming. Sommer (1981), Kacprzyk and Taniewski (1982) address the problem of incorporating fuzziness into the modeling an inventory control problems. Sommer (1981) applies fuzzy dynamic programming given by Bellman and Zadeh (1970). From a global perspective of top management, Kacprzyk and Staniewski (1982) regarded optimal company determination in a fuzzy environment with fuzzy constraints.In this manner so many researchers developed inventory models in fuzzy environment. Recently, Mondal and Maiti (2002) formulated fuzzy inventory models as fuzzy non-linear decision making issues and solved by Genetic algorithm method and FNLP method. Following Sakawa (1983, 1986), interactive fuzzy approach in general form was initiated for inventory control system by Dey *et al.* (2005). Rezaei and Davoodi (2005) explored fuzzy multi-item inventory model with restrictions on total cost of production, total storage space and number of orders. They had given solution under fuzzy objective of cost minimization by using Genetic algorithm. Dutta *et al.* (2005) proposed a single-period inventory model with demand as fuzzy random variable. Umap and Bajaj (2007) constructed multi item EOQ model for decaying item in fuzzy environment. In this model total cost, warehouse space, deterioration rate are considered to be vague and bothlinear and non-linear membership functions represent the vagueness of these parameters.Roy *et al.* (2008) built an economic ordered quantity model for decaying items in a fuzzy environment where inventory costs, storage space and budget are deemed to be vague and vagueness is represented by fuzzy numbers. Panda *et al.* (2009) developed two economic production quantity inventory models in fuzzy environments. The first model is designed with fuzzy goal and fuzzy storage area constraint and the second model with unit cost as fuzzy and possibility restriction on storage space. Bera *et al.* (2012) developed a model of inventory with a vague horizon of time. De and Sana (2013) built backorder economic ordered quantitymodel by considering decision variables in fuzzy environment. Above discussed inventory models developed in fuzzy environment for single warehouse facility. But as discussed earlier, to accommodate large volume of items two warehouse facility is needed. Following some researchers studied this situation thoroughly and developed inventory models for two warehouse facility in fuzzy environment

Maiti (2006) developed first multi-item fuzzy inventory model for two warehouse facilitywith purchase cost, investment amount and storehouse capacityare imprecise. Maiti (2007) constructed multi-item fuzzy two storage facilities inventory model with stock dependent demand. Maiti *et al.* (2007) addressed two warehouse multi - item inventory model with linearly time dependent demand under crisp, stochastic and fuzzy-stochastic environments. Roy (2007) proposed two storage facility inventory modelwith imprecise deterioration rate and stock dependent demand. Rong (2008) developed fuzzy two storage facility inventory model inventory policy for a decaying item with imprecise lead-time, partially/fully backlogged shortages and price dependent demand. Maiti (2008) formed multi-item two storage facilities fuzzy inventory model with stock-dependent demand under inflation and time value of money. He considered purchase cost, budget and warehouse capacity are vague. Umap (2010) developed two storage facility fuzzy inventory models with holding cost and deterioration cost are considered as fuzzy. Singh (2011) built a two storage facility fuzzy inventory model under the circumstancesof acceptable delay in payments. Maiti (2011) described the possibility and necessity representations of fuzzy inequality restrictions and using this for a finite period of time multi-item two warehouse inventory model for manufacturing process with fuzzy restrictions has been formulated. Yadav (2012) formed multi-item two warehouse facility inventory modelwith stock dependent demand under inflation and time value of money for decaying items with purchase cost, budget and warehouse capacity are vague.Soni (2015) explores two-warehouse inventory model under conditionally acceptable delay in payment with fuzzy demand and deteriorationrate.

In this chapter two warehouse inventory models have been developed with ramp type demand and constant demand. Inventory models developed in this chapter have objective of maximize profit. First inventory model is developed for instantaneous deteriorating items while second model is developed for non-instantaneous deteriorating items. Using Numerical examples models are illustrated and sensitivity analysis is provided.

2.2 Some Fuzzy Inventory models for decaying items with two warehouse facility:

2.2.1 Model I: Two warehouse fuzzy inventory model for instantaneous decaying items with Ramp type demand:

Model discussed in this section is an attempt to study idea on the basis of the model developed and applied by Saha *et al.* (2012). The model is multi-item profit maximization with ramp type demand under imprecise space and budget constraint with two storage facility.

In this model, it is considered there are two warehouses, one is situated at market place named as W_1 and other is little away from market place named as W_2. Demand of the items is met using the inventory of W_1 and which are filled up from W_2 in continuous release pattern.

2.2.1.1: Assumptions

- Lead time is zero.
- Replenishment rate is infinite but replenishment size is finite.
- Shortages are not allowed.

2.2.1.2: Notations

Qi: Maximum number of units stored in $W_1$for i^{th} item

Si: Maximum number of units stored in $W_2$for i^{th} item.

$C_{1i:}$ Holding costs per unit item, per unit time for $W_1$for i^{th} item.

C_{2i}: Holding costs per unit item, per unit time for $W_2$for i^{th} item.

C_{d1i}: Decaying costs per unit item, per unit time for W_1

C_{d2i}: Decaying costs per unit item, per unit time for W_2

$Q_{1i}(t)$: On hand inventory at time t in $W_1$for i^{th} item.

$Q_{2i}(t)$: On hand inventory at time t in $W_2$for i^{th} item.

θ_{1i}: Deterioration rate in W_1 for i^{th} item.

θ_{2i}: Deterioration rate in W_2 for i^{th} item.

Ti: Total cycle length.

P_i: Selling price per item for i^{th} item

C_i: Purchasing price per item for i^{th} item.

w_i: Space required per item of i^{th} item.

W: Total Warehouse space available

B: Total Budget Available for purchasing item

(Wavy bar (~) represents the fuzzification of the parameters)

2.2.1.3: Crisp Model

Demand type: Ramp Type Demand

Demand Function:

$$D(t) = \begin{cases} Do * t & t \leq \mu \\ Do * \mu & t \geq \mu \end{cases}$$

Where Do and μ are constants. Here for time period (0,μ) demand is increasing function of time and for time period (μ,T) it remains constant.

Let the differential equation describing state of inventory in Warehouse (W_1) in time period ($0,\mu_i$) is as follows

$$\frac{dQ_{1i}(t)}{dt} + \theta_{1i} * Q_{1i}(t) = -D_{oi} * t, 0 \leq t \leq \mu_i \text{ - (1)}$$

And differential equation describing state of inventory in Warehouse (W_1) in time period ($\mu i,T_i$) is as follows

$$\frac{dQ_{1i}(t)}{dt} + \theta_{1i} * Q_{1i}(t) = -D_{oi} * \mu_i, \mu_i \leq t \leq T_i \text{ - (2)}$$

Solving equation (1) by using condition Q_{1i} (0) =Qi

$$Q_{1i}(t) = -D_{oi} * \left[\frac{t}{\theta_{1i}} - \frac{1}{\theta_{1i}^2} \right] + \left(Q_i - \frac{D_{oi}}{\theta_{1i}^2} \right) * e^{-\theta_{1i} * t}, 0 \leq t \leq \mu_i \text{ - (3)}$$

Solving equation (2) by using condition Q_{1i} (T_i) =0

$$Q_{1i}(t) = -\frac{D_{oi} * \mu_i}{\theta_1} + \left(\frac{D_{oi} * \mu_i}{\theta_1} \right) * e^{-\theta_1 * (T_i - t)}, \mu_i \leq t \leq T_i \text{ - (4)}$$

At t=μ_i, Q_{1i} remains same for equation (3) and (4)

$$\frac{D_{oi}}{\theta_{1i}^{\ 2}} + \left(Q_i - \frac{D_{oi}}{\theta_{1i}^{\ 2}} \right) * e^{-\theta_{1i} * \mu_i} - \frac{D_{oi} * \mu_i}{\theta_{1i}} * e^{-\theta_{1i} * (T_i - \mu_i)} = 0$$

Let the differential equation describing state of inventory in Warehouse (W_2) in time period (0,μi) is as follows

$$\frac{dQ_{2i}(t)}{dt} + \theta_{2i} * Q_{2i}(t) = -D_{oi} * t, 0 \leq t \leq \mu_i \text{ - (5)}$$

And differential equation describing state of inventory in Warehouse (W_2) in time period ($\mu i,T_i$) is as follows

$$\frac{dQ_{2i}(t)}{dt} + \theta_{2i} * Q_{2i}(t) = -D_{oi} * \mu_i, \mu_i \leq t \leq t_{1i} \text{ - (6)}$$

Solving equation (5) by using condition $Q_{2i}(0)$ =Si

$$Q_{2i}(t) = -D_{oi} * \left[\frac{t}{\theta_{2i}} - \frac{1}{\theta_{2i}^2} \right] + \left(S_i - \frac{D_{oi}}{\theta_{2i}^2} \right) * e^{-\theta_{2i} * t}, 0 \le t \le \mu_i \quad -(7)$$

Solving equation (6) by using condition at $t=\mu_i$, Q_{2i} remains same for equation (5) and(6)

$$Q_{2i}(t) = -\frac{D_{oi} * \mu_i}{\theta_{2i}^2} + \frac{D_{oi} * e^{\theta_{2i} * (\mu_i - t)}}{\theta_{2i}^2} + \left(S_i - \frac{D_{oi}}{\theta_{2i}^2} \right) * e^{-\theta_{2i} * t}, \mu_i \le t \le t_{1i} \quad -(8)$$

At $t=t_{1i}$, $Q_{2i}(t)=0$ and using series form of exponential term and ignoring second and higher terms

$$t_{1i} = \frac{1}{\theta_{2i}}$$

Let total holding costs (HC) for W_1 and W_2 given by following equations

$$HC_{W_1} = C_{1i} * \left(\int_{\mu_i}^{T_i} Q_{1i}(t) + \int_0^{\mu_i} Q_{1i}(t) \right)$$

$$HC_{W_1} = C_{1i} * \left(\begin{aligned} & \frac{D_{oi} * \mu_i^2}{2 * \theta_{1i}} + \frac{D_{oi} * \mu_i}{\theta_{1i}^2} + \left(Q_i - \frac{D_{oi}}{\theta_{1i}^2} \right) * \left(\frac{\left(1 - e^{\theta_{1i} * \mu_i} \right)}{\theta_{1i}} \right) - \frac{D_{oi} * \mu_i * T_i}{\theta_{1i}} \\ & + \left(\frac{D_{oi} * \mu_i}{\theta_{1i}^2} \right) * \left(e^{\theta_{1i} * (T_i - \mu_i)} - 1 \right) \end{aligned} \right)$$

$$HC_{W_2} = C_{2i} * \left(\int_{\mu_i}^{t_{1i}} Q_{2i}(t) + \int_0^{\mu_i} Q_{2i}(t) \right)$$

$$HC_{W_2} = C_{2i} * \left(\begin{aligned} & -\frac{D_{oi} * \mu_i^2}{2 * \theta_{2i}} + \frac{D_{oi} * \mu_i}{\theta_{2i}^2} + \left(S_i - \frac{D_{oi}}{\theta_{2i}^2} \right) * \left(\frac{\left(1 - e^{\theta_{2i} * \mu_i} \right)}{\theta_{2i}} \right) - \frac{D_{oi} * \mu_i * t_{1i}}{\theta_{2i}} + \frac{D_{oi} * \mu_i^2}{\theta_{2i}} \\ & + \left(\frac{D_{oi}}{\theta_{2i}^3} \right) * \left(1 - e^{\theta_{2i} * (\mu_i - t_{1i})} \right) + \left(S_i - \frac{D_{oi}}{\theta_{2i}^2} \right) * \left(\frac{e^{-\theta_{2i} * \mu_i} - e^{-\theta_{2i} * t_{1i}}}{\theta_{2i}} \right) \end{aligned} \right)$$

Hence the inventory problem is described as follows

$$\mathrm{Max}PF(Q_i) = \sum_{i=1}^{n} PF(Q_i) = \sum_{i=1}^{n} \begin{pmatrix} (P_i - C_i)*(Q_i + S_i) \\ -(C_{2i} + C_{d_{2i}} * \theta_{2i})*\left(\int_{\mu_i}^{t_{1i}} Q_{2i}(t) + \int_{0}^{\mu_i} Q_{2i}(t) \right) \\ -(C_{1i} + C_{d_{1i}} * \theta_{1i})*\left(\int_{\mu_i}^{T_i} Q_{1i}(t) + \int_{0}^{\mu_i} Q_{1i}(t) \right) \end{pmatrix}$$

such that

$$\frac{D_{oi}}{\theta_{1i}^{2}} + \left(Q_i - \frac{D_{oi}}{\theta_{1i}^{2}} \right) * e^{-\theta_{1i}*\mu_i} - \frac{D_{oi} * \mu_i}{\theta_{1i}} * e^{-\theta_{1i}*(T_i - \mu_i)} = 0$$

$$t_{1i} = \frac{1}{\theta_{2i}}$$

$$\sum P_i * (Q_i + S_i) \le B$$

$$\sum w_i * (Q_i + S_i) \le W$$

2.2.1.4 Fuzzy Model:

In above crisp model holding costs in both warehouses, Budget and warehouse space are considered to be vague and imprecise and there vagueness is represented by fuzzy numbers. Then the above crisp model converted into fuzzy model as follows

$$\max \tilde{P}F = \sum_{i=1}^{n} PF(Q_i)$$

Subject to

$$\sum P_i * (Q_i + S_i) \le \tilde{B}$$

$$\sum w_i * (Q_i + S_i) \le \tilde{W}$$

$$\frac{D_{oi}}{\theta_{1i}^{2}} + \left(Q_i - \frac{D_{oi}}{\theta_{1i}^{2}} \right) * e^{-\theta_{1i}*\mu_i} - \frac{D_{oi} * \mu_i}{\theta_{1i}} * e^{-\theta_{1i}*(T_i - \mu_i)} = 0$$

$$t_{1i} = \frac{1}{\theta_{2i}}$$

$$Q_i \ge 0, i = 1, 2, ..., n$$

Let $\mu_{PF}, \mu_W, \mu_B, \mu_{c_{1i}}, \mu_{c_{2i}}$ be the linear membership functions represents PF, W, B, C_{1i}, C_{2i} respectively and are given by

$$\mu_{PF} = \begin{cases} 0 & PF \le Co - P_{PF} \\ 1 - \dfrac{Co - PF}{P_{PF}} & Co - P_{PF} \le PF \le Co \\ 1 & PF \ge Co \end{cases}$$

$$\mu_{W} = \begin{cases} 0 & \sum_{i=1}^{n} w_i * (Q_i + S_i) \ge W + P_W \\ 1 - \dfrac{\sum_{i=1}^{n} w_i * (Q_i + S_i) - W}{P_W} & W \le \sum_{i=1}^{n} w_i * (Q_i + S_i) \le W + P_W \\ 1 & \sum_{i=1}^{n} w_i * (Q_i + S_i) \le W \end{cases}$$

$$\mu_{B} = \begin{cases} 0 & \sum p_i * (Q_i + S_i) \ge B + P_B \\ 1 - \dfrac{\sum p_i * (Q_i + S_i) - B}{P_B} & B \le \sum p_i * (Q_i + S_i) \le B + P_B \\ 1 & \sum p_i * (Q_i + S_i) \le B \end{cases}$$

$$\mu_{C_{1i}} = \begin{cases} 0 & C_{1i} \ge C_{01i} + P_{C_{1i}} \\ 1 - \dfrac{C_{1i} - C_{01i}}{P_{C_{1i}}} & C_{01i} \le C_{1i} \le C_{01i} + P_{C_{1i}} \\ 1 & C_{1i} \le C_{01i} \end{cases}$$

$$\mu_{C_{2i}} = \begin{cases} 0 & C_{2i} \ge C_{02i} + P_{C_{2i}} \\ 1 - \dfrac{C_{2i} - C_{02i}}{P_{C_{02i}}} & C_{02i} \le C_{2i} \le C_{02i} + P_{C_{2i}} \\ 1 & C_{2i} \le C_{02i} \end{cases}$$

Using fuzzy non-linear programming technique the solution of fuzzy inventory model is transformed to

$$\text{Max} = \alpha$$
Subject to

$$1-\frac{Co-PF}{P_{PF}} \geq \alpha$$

$$1-\frac{\sum_{i=1}^{n} w_i *(Q_i+S_i)-W}{P_W} \geq \alpha$$

$$1-\frac{\sum P_i *(Q_i+S_i)-B}{P_B} \geq \alpha$$

$$\frac{D_{oi}}{\theta_{1i}^{2}}+\left(Q_i-\frac{D_{oi}}{\theta_{1i}^{2}}\right)*e^{-\theta_{1i}*\mu_i}-\frac{D_{oi}*\mu_i}{\theta_{1i}}*e^{-\theta_{1i}*(T_i-\mu_i)}=0$$

$$t_{1i}=\frac{1}{\theta_{2i}}$$

$$Q_i \geq 0,\ i=1,2,....,n$$

Where

$$PF=\sum_{i=1}^{n}\left(\begin{array}{l}(P_i-C_i)*(Q_i+S_i)-(C_{2i}+(1-\alpha)P_{C_{2i}}+C_{d_{2i}}*\theta_{2i})*\left(\int_{\mu_i}^{t_{1i}} Q_{2i}(t)+\int_{0}^{\mu_i} Q_{2i}(t)\right)- \\ (C_{1i}+(1-\alpha)P_{C_{1i}}+C_{d_{1i}}*\theta_{1i})*\left(\int_{\mu_i}^{T_i} Q_{1i}(t)+\int_{0}^{\mu_i} Q_{1i}(t)\right)\end{array}\right)$$

2.2.1.5 Numerical Example:

a) Crisp Model:

Input:

$\theta_{11}=\theta_{12}=0.4$, $\theta_{21}=\theta_{22}=0.6$, , $P_1= P_2=10$ Rs., $C_1=7$ Rs., $C_2=6$ Rs., $S_1=35$, $S_2=30$, $T_1=2$, $T_2=2$, $D_{o1}=100$,$D_{o2}=110$, B=2600Rs. ,W=250 Sq. Ft ,$w_1=0.80$ Sq. Ft,$w_2=0.60$ Sq. Ft,$C_{11}= C_{12} = 2.4$ Rs., $C_{21}= C_{21}= 2.2$ Rs., $C_{d11}= C_{d12} =1.1$Rs.,$C_{d21}= C_{d22} =1$Rs.

Output:

PF=1437.14 Rs., $Q_1= 79.53$, $Q_2=233.95$,$\mu_1=0.27$yr. ,$\mu_2=0.81$yr., $t_{11}=1.66$ yr., $t_{12}=1.66$ yr.

b) Fuzzy Model

Input:

$\theta_{11}=\theta_{12}=0.4$, $\theta_{21}=\theta_{22}=0.6$, $P_1= P_2=10$ Rs., $C_1=7$ Rs., $C_2=6$ Rs., $S_1=35$, $S_2=30$,

$T_1=2$ yr., $T_2=2$ yr., $D_{o1}=100$,$D_{o2}=110$,B=2600Rs. ,W=250 Sq. Ft ,$w_1=0.80$ Sq. Ft,$w_2=0.60$ Sq. Ft ,Co=1450,$P_{PF}=200$, $C_{11}= 2.2$,$C_{12} = 2.4$ Rs., $C_{21}= C_{21}= 2.4$ Rs., $C_{d11}= C_{d12} =1.1$Rs.,$C_{d21}= C_{d22} =1$Rs.,$P_W=20$,$P_B=300$ Rs,$P_{C11}=P_{C12}=0.2$ Rs.

Output:

$\alpha=0.95$, PF=1440.69 Rs., $Q_1=80.37$, $Q_2=234.36$,$\mu_1=0.27$yr. ,$\mu_2=0.81$yr. , $t_{11}=1.66$ yr., $t_{12}=1.66$ yr.

2.2.1.6 Sensitivity Analysis:

Table 2.1: Effect of change of values in θ_1

θ_1	θ_2	PF	Q_1	Q_2	μ_1	μ_2
0.2	0.6	2559.898	100.9087	205.4551	0.45	0.95
0.3	0.6	1812.625	85.78668	225.6178	0.33	0.91
0.4	0.6	1437.14	79.53	233.95	0.27	0.81

Table 2.2: Effect of change of values in D_{oi}

D_{01}	D_{02}	PF	Q_1	Q_2	μ_1	μ_2
100	110	1437	79	233	0.2	0.8
120	110	1430	81	232	0.2	0.8
140	110	1455	87	224	0.2	0.8
150	140	1464	90	220	0.2	0.8

Table 2.3: Effect of change of values in Co

C_0	α	PF	Q_1	Q_2	μ_1	μ_2
1450	0.9539965	1440.690	80.37390	234.3683	0.2751506	0.8173476
1470	0.8735216	1444.409	81.29299	234.7713	0.2784672	0.8190959
1490	0.7961596	1448.768	82.36566	235.2751	0.2823437	0.8212841
1500	0.7573719	1450.931	82.90313	235.5282	0.2842883	0.8223844

Table 2.4: Effect of change of values in P_{PF}

C_0	P_{PF}	α	PF	Q_1	Q_2	μ_1	μ_2
1450	230	0.9584634	1440.347	80.29268	234.3276	0.2748577	0.8171716
1450	250	0.9609881	1440.154	80.24677	234.3047	0.2746922	0.8170721
1450	270	0.9632232	1439.982	80.20613	234.2844	0.2745456	0.8169841

Table 2.5: Effect of change of values in W

W	P_W	α	PF	Q_1	Q_2	μ_1	μ_2
220	20	0.4645146	1341.297	61.26196	221.7045	0.2071512	0.7634046
230	20	0.6359257	1376.102	67.43640	225.8533	0.2289227	0.7808725
240	20	0.7980883	1409.057	73.75453	230.0417	0.2513929	0.7987065
250	20	0.9539965	1440.690	80.37390	234.3683	0.2751506	0.8173476

Table 2.6: Effect of change of values in Pw

W	P_W	α	PF	Q_1	Q_2	μ_1	μ_2
250	10	0.9466687	1439.205	80.04873	234.1572	0.2739782	0.8164329
250	15	0.9506055	1440.003	80.22327	234.2705	0.2746074	0.8169239
250	20	0.9539965	1440.690	80.37390	234.3683	0.2751506	0.8173476
250	25	0.9569481	1441.287	80.50526	234.4535	0.2756243	0.8177171

Table 2.7: Effect of change of values in C_{1i}

C_{1i}	α	PF	Q_1	Q_2	μ_1	μ_2
2.2	0.8900503	1444.733	78.24399	239.3397	0.2674815	0.8390553
2.3	0.9213129	1442.993	79.35173	236.8206	0.2714671	0.8280164
2.4	0.9539965	1440.690	80.37390	234.3683	0.2751506	0.8173476
2.5	0.9879975	1437.878	81.31606	231.9787	0.2785506	0.8070238

Table 2.8: Effect of change of values in P_{C1i}

P_{C1i}	α	PF	Q_1	Q_2	μ_1	μ_2
0.2	0.9539965	1440.690	80.37390	234.3683	0.2751506	0.8173476
0.3	0.9554829	1440.575	80.41750	234.2606	0.2753078	0.8168809
0.4	0.9568784	1440.466	80.45823	234.1598	0.2754547	0.8164440
0.5	0.9581910	1440.364	80.49635	234.0652	0.2755922	0.8160343

2.2.1.7 General observations from sensitivity analysis:

Table 2.1 shows the effect of deterioration rate, it shows as deterioration rate increases profit decreases significantly. Table 2.2 shows, as parameter of demand Doi increases, profit increases significantly. In table 2.3 it is observed that as a goal of profit C0 increases, profit increases significantly but the level of satisfaction α decreases significantly. From table 2.4 it is seen that as the aspiration level of PF increases, profit decreases and level of satisfaction α increases. From table 2.5, it is pointed that as warehouse space increases, profit as well as the level of satisfaction α increases significantly. From table 2.6, it is observed that as aspiration level of W increases, profit as well as level of significance α increases significantly. Tables 2.7 and 2.8 showed that as holding cost and its aspiration level increases, profit decreases and level of satisfaction α increases significantly.

2.2.2 Model II: Two warehouse fuzzy inventory model for non-instantaneous decaying items with constant demand:

This model is developed specially for non-instantaneous decaying items. Decaying or decaying is the most important aspect of the inventory replenishment policy models. In items such as fruits, vegetables, pharmaceuticals, volatile liquids and others deterioration is observed. Inventory models developed for such items considered that the deterioration of an item is starts from its arrival in inventory, but in practice it is seen that most of the commodities have a short maintenance period of fresh quality, during which there is almost no spoilage. This aspect was first introduced in the inventory model by Wu *et al.* (2006), Ouyang *et al.* (2006) and referred to as "non - instant deterioration."

Here also for the same two warehousing situation, same assumptions and notations as in model 3.2.1 are considered to develop inventory model in fuzzy environment. Fuzzy parameters are represented by triangular fuzzy number and defuzzified by centroid method.

2.2.2.1: Crisp Model

Demand type: Constant Demand

Demand Function:

Demand = α_i

Here it is considered that for time interval $(0,\mu_i)$ product remains in fresh condition without deterioration and onward in time interval (μ_i,T_i) items get deteriorating.

Let the differential equation describing state of inventory in Warehouse (W_1) in time period $(0,\mu_i)$ is as follows

$$\frac{dQ_{1i}(t)}{dt} = -\alpha_i, 0 \le t \le \mu_i \quad -(1)$$

And the differential equation describing state of inventory in Warehouse (W_1) in time period (μ_i,T_i) is as follows

$$\frac{dQ_{1i}(t)}{dt} + \theta_{1i} * Q_{1i}(t) = -\alpha_i, \mu_i \le t \le T_i \quad -(2)$$

Solving equation (1) by using condition Q_{1i} (0) =Qi

$$Q_{1i}(t) = -\alpha_i * t + Q_i, 0 \le t \le \mu_i \quad -(3)$$

Solving equation (2) by using condition Q_{1i} (T_i) =0

$$Q_{1i}(t) = -\alpha_i + \alpha_i * e^{\theta_{1i}*(T_i - t)}, \mu_i \le t \le T_i \quad -(4)$$

At t=μ_i; Q_{1i} remains same for equation (3) and (4)

$$-\alpha_i * \mu_i + Q_i + \alpha_i - \alpha_i * e^{\theta_1 *(T_i - \mu_i)} = 0$$

Let the differential equation describing state of inventory in Warehouse (W_2) in time period (0, μ_i) is as follows

$$\frac{dQ_{2i}(t)}{dt} = -\alpha_i, 0 \le t \le \mu_i \quad -(5)$$

And the differential equation describing state of inventory in Warehouse (W_2) in time period (μ_i,T_i) is as follows

$$\frac{dQ_{2i}(t)}{dt} + \theta_{2i} * Q_{2i}(t) = -\alpha_i, \mu_i \le t \le t_{1i} \quad -(6)$$

Solving equation (5) by using condition Q_{2i}(0) =Si

$$Q_{2i}(t) = -\alpha_i * t + S_i, 0 \le t \le \mu_i \quad -(7)$$

Solving equation (6) by using condition at t=μ_i; Q_{2i} remains same for equation (5) and (6)

$$Q_{2i}(t) = -\frac{\alpha_i}{\theta_{2i}} + \left(\frac{\alpha_i}{\theta_{2i}} + S_i - \alpha_i * \mu_i\right) * e^{\theta_{2i}*(\mu_i - t)}, \mu_i \le t \le t_{1i} \quad -(8)$$

Using t=t_{1i}, Q_{2i}=0 in above equation gives

$$t_{1i} = \mu_i - \left(\frac{1}{\theta_{2i}}\right) * \log\left(1 + \frac{\theta_{2i} * S_i}{\alpha_i} - \theta_{2i} * \mu_i\right)$$

Using series form of log function and ignoring higher of θ_{2i} gives

$$t_{1i} = 2*\mu_i - \frac{S_i}{\alpha_i}$$

Let total holding costs (HC) for W_1 and W_2 given by following equations

$$HC_{W_1} = C_{1i} * \left(\int_{\mu_i}^{T_i} Q_{1i}(t) + \int_{0}^{\mu_i} Q_{1i}(t) \right)$$

$$HC_{W_1} = C_{1i} * \begin{pmatrix} -\alpha_i * \mu_i^2 + Q_i * \mu_i - \alpha_i * T_i + \alpha_i * \mu_i \\ + \frac{\alpha_i}{\theta_1} * \left(e^{\theta_1 * (T_i - \mu_i)} - 1 \right) \end{pmatrix}$$

$$HC_{W_2} = C_{2i} * \left(\int_{\mu_i}^{t_{1i}} Q_{2i}(t) + \int_{0}^{\mu_i} Q_{2i}(t) \right)$$

$$HC_{W_2} = C_{2i} * \begin{pmatrix} -\frac{\alpha_i * \mu_i^2}{2} + S_i * \mu_i + \frac{\alpha_i * \mu_i}{\theta_{2i}} - \frac{\alpha_i * t_{1i}}{\theta_{2i}} + \\ \frac{1}{\theta_{2i}} * \left(\frac{\alpha_i}{\theta_{2i}} + S_i - \alpha_i * \mu_i \right) * \left(1 - e^{\theta_{2i} * (\mu_i - t_{1i})} \right) \end{pmatrix}$$

Hence crisp inventory model is described as follows

$$MaxPF = \sum_{i=1}^{n} \begin{pmatrix} (P_i - C_i) * (Q_i + S_i) - (C_{2i} + C_{d_2} * \theta_{2i}) * \begin{pmatrix} -\frac{\alpha_i * \mu_i^2}{2} + S_i * \mu_i + \frac{\alpha_i * \mu_i}{\theta_{2i}} - \frac{\alpha_i * t_{1i}}{\theta_{2i}} \\ + \frac{1}{\theta_{2i}} * \left(\frac{\alpha_i}{\theta_{2i}} + S_i - \alpha_i * \mu_i \right) * \left(1 - e^{\theta_{2i} * (\mu_i - t_{1i})} \right) \end{pmatrix} \\ -(C_{1i} + C_{d_1} * \theta_{1i}) * \begin{pmatrix} -\alpha_i * \mu_i^2 + Q_i * \mu_i - \alpha_i * T_i + \alpha_i * \mu_i \\ + \frac{\alpha_i}{\theta_{1i}} * \left(e^{\theta_{1i} * (T_i - \mu_i)} - 1 \right) \end{pmatrix} \end{pmatrix}$$

such that

$$t_{1i} = 2*\mu_i - \frac{S_i}{\alpha_i}$$

$$-\alpha_i * \mu_i + Q_i + \alpha_i - \alpha_i * e^{\theta_1 * (T_i - \mu_i)} = 0$$

2.2.2.2 Fuzzy Model:

In the above developed crisp model, it is assumed that all the parameters are assumed to be fixed or could be predicted with certainty, but in practical situations, they will fluctuate little. Therefore parameters of model instead of assumed to be constant, are taken as fuzzy.

Here deterioration rate, holding costs in both warehouses, purchasing cost, μ are considered as fuzzy and are represented by triangular fuzzy numbers. Then the above crisp model converted into fuzzy model as follows

$$
\mathrm{Max}\,\tilde{PF} = \sum_{i=1}^{n}\left(\begin{array}{l}
(P_i - \tilde{C}_i)*(Q_i + S_i) \\
-(\tilde{C}_{2i} + C_{d_{2i}} * \tilde{\theta}_{2i})*\left(\begin{array}{l}
-\dfrac{\alpha_i * \tilde{\mu}_i^{2}}{2} + S_i * \tilde{\mu} + \dfrac{\alpha_i * \tilde{\mu}_i}{\tilde{\theta}_{2i}} - \dfrac{\alpha_i * t_{1i}}{\tilde{\theta}_{2i}} \\
+\dfrac{1}{\tilde{\theta}_{2i}} * \left(\dfrac{\alpha_i}{\tilde{\theta}_{2i}} + S_i - \alpha_i * \tilde{\mu}_i\right) * \left(1 - e^{\tilde{\theta}_{2i}*(\mu_i - t_{1i})}\right)
\end{array}\right) \\
-(\tilde{C}_{1i} + C_{d_{1i}} * \tilde{\theta}_{1i})*\left(\begin{array}{l}
-\alpha_i * \tilde{\mu}_i^{2} + Q_i * \tilde{\mu}_i - \alpha_i * T_i + \alpha_i * \tilde{\mu}_i \\
+\dfrac{\alpha_i}{\tilde{\theta}_{1i}} * \left(e^{\tilde{\theta}_{1i}*(T_i - \tilde{\mu}_i)} - 1\right)
\end{array}\right)
\end{array}\right)
$$

Such that

$$t_{1i} = 2*\tilde{\mu}_i - \frac{S_i}{\alpha_i}$$

$$-\alpha_i * \tilde{\mu}_i + Q_i + \alpha_i - \alpha_i * e^{\tilde{\theta}_{1i}*(T_i - \tilde{\mu}_i)} = 0$$

Defuzzification by Centroid method:

In real life, it is very difficult to consider the deterioration rates to be constant in W_1 and W_2 over a total time period Ti. Rather than considering deterioration rate as constant, it is easy to locate it in interval.

Let deterioration rate in W_1 is located in an interval $(\theta_1 - \Delta_1, \theta_1 + \Delta_2)$ where

$0 < \Delta_1 < \theta_1$ and $\Delta_1 * \Delta_2 > 0$

And deterioration rate in W_2 is located in an interval $(\theta_2 - \Delta_3, \theta_2 + \Delta_4)$ where $0 < \Delta_3 < \theta_2$ and $\Delta_3 * \Delta_4 > 0$ here $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ will be decided by decision maker and θ_1, θ_2 are known numbers.

Therefore corresponding to the intervals $(\theta_1 - \Delta_1, \theta_1 + \Delta_2)$ and $(\theta_2 - \Delta_3, \theta_2 + \Delta_4)$ we set the following fuzzy numbers.

$\tilde{\theta}1 = (\theta_1 - \Delta_1, \theta_1, \theta_1 + \Delta_2), 0 < \Delta_1 < \theta_1, \Delta_1 * \Delta_2 > 0$

$\tilde{\theta}2 = (\theta_2 - \Delta_3, \theta_2, \theta_2 + \Delta_4), 0 < \Delta_3 < \theta_2, \Delta_3 * \Delta_4 > 0$

Then the Centroid of $\tilde{\theta}_1$ is,

$$C(\tilde{\theta}_1) = \theta_1 + \frac{1}{3}(\Delta_2 - \Delta_1)$$

where $C(\tilde{\theta}_1)$ is the deterioration rate in fuzzy sense based on centroid. If $\Delta_1 = \Delta_2$ then the deterioration rate in the fuzzy sense is exactly the same as the crisp deterioration rate.

Similarly holding costs of W_1 and W_2, purchasing cost, μ_i may fluctuate little in a period T. Suppose they lies in the intervals $(c_{1i} - \Delta_{5i}\ , c_{1i} + \Delta_{6i})$, $(c_{2i} - \Delta_{7i}\ ,\ c_{1i} + \Delta_{8i})$, $(c_i - \Delta_{9i}\ ,\ c_i + \Delta_{10i})$, $(\mu - \Delta_{11i}\ ,\ \mu + \Delta_{12i})$ respectively. Similarly, corresponding to the above intervals we set the following fuzzy numbers.

$\tilde{c}_{1i} = (c_{1i}-\Delta_{5i}, c_{1i}, c_{1i}+\Delta_{6i}),\ 0<\Delta_{5i}<c_{1i}\ \ , 0 <\Delta_{5i}\Delta_{6i}$

$\tilde{c}_{2i} = (c_{2i}-\Delta_{7i}, c_{2i}, c_2+\Delta_{8i}),\ 0<\Delta_{7i}<c_{2i}\ \ , 0 <\Delta_{7i}\Delta_{8i}$

$\tilde{c}_i = (c_i-\Delta_{9i}, c_i, c_i+\Delta_{10i}), 0<\Delta_{9i}<c_i\ \ , 0 <\Delta_{9i}\Delta_{10i}$

$\tilde{\mu}_i = (\mu_i-\Delta_{11}, \mu_i, \mu_i-\Delta_{12}), 0 <\Delta_{11}<\mu_i\ \ , 0 <\Delta_{11}\Delta_{12}$

Then by centroid of $\tilde{c}_{1i}, \tilde{c}_{2i}, \tilde{c}_i$, and μ are

$$C(\tilde{c}_{1i}) = c_{1i} + \frac{1}{3}(\Delta_{6i} - \Delta_{5i})$$

$$C(\tilde{c}_{2i}) = c_{2i} + \frac{1}{3}(\Delta_{8i} - \Delta_{7i})$$

$$C(\tilde{c}_i) = c_i + \frac{1}{3}(\Delta_{10i} - \Delta_{9i})$$

$$C(\tilde{\mu}_i) = \mu_i + \frac{1}{3}(\Delta_{12i} - \Delta_{11i})$$

Then fuzzy profit is obtained by just replacing above centroids in fuzzy model.

2.2.2.3 Numerical Example:

a) Crisp Model

Input:

$P_1=P_2=10$ Rs., $C_1=7$ Rs., $C_2=10$ Rs., $c_{11}=c_{12}=2.2$ Rs.,$c_{21}=c_{22}=2.4$ Rs., $\theta_{11}= \theta_{12}=0.4$, $\theta_{21}=\theta_{22}=0.6$, $\mu_1=0.7$, $\mu_2=0.75$,$S_1=35$, $S_2=30$, $T1=T_2=2$ yr., $\alpha_1=100$, $\alpha_2=110$

Output:

$Q_1=138.20$, $Q_2=153.85$, PF=1609.56 Rs., $t_{11}=1.05$ yr., $t_{12}=1.22$ yr.

b) Fuzzy Model

Input:

$P_1=P_2=10$ Rs., $C_1=7$ Rs., $C_2=10$ Rs., $c_{11}=c_{12}=2.2$ Rs.,$c_{21}=c_{22}=2.4$ Rs., $\theta_{11}= \theta_{12}=0.4$, $\theta_{21}=\theta_{22}=0.6$,$\mu_1=0.7$, $\mu_2=0.75$,$S_1=35$, $S_2=30$, $T1=T_2=2$ yr., $\alpha_1=100$, $\alpha2=110$, $\Delta_1=0.05$,$\Delta_2=0.0$,$\Delta_3=0.05$,$\Delta_4=0.0$,$\Delta_{51}=\Delta_{52}=0.10$, $\Delta_{61}=\Delta_{62}=0.65$,$\Delta_{71}=\Delta_{72}=0.10$,$\Delta_{81}=\Delta_{82}=0.65$, $\Delta_{91}=\Delta_{92}=0.5$,$\Delta_{101}=2.5$, $\Delta_{102}=2.5$,$\Delta_{111}= \Delta_{112}=0.002$,$\Delta_{121}=\Delta_{122}=0.07$

Output:

$Q_1=137.54$, $Q_2=153.09$, PF=1715.06 Rs., $t_{11}=1.004$ yr., $t_{12}=1.18$ yrs

2.2.2.4: Sensitivity Analysis

Δ7	**Δ8**	**Δ5**	**Δ6**	**Δ3**	**Δ4**	**Δ1**	**Δ2**	**Δ9**	**Δ10**	**Δ11**	**Δ12**	**PF**
0.75	0.95	0.85	1	0.001	0.003	0.005	0.007	1	1.5	0.008	0.01	**1563.686**
0.25	0.35	0.33	0.43	0.07	0.09	0.08	0.1	1.5	2	0.2	0.24	**1546.906**
0.75	0.95	0.85	1	0.001	0.003	0.005	0.007	2	2.5	0.008	0.01	**1563.686**
0.25	0.35	0.33	0.43	0.05	0.06	0.07	0.08	2	2.5	0.065	0.075	**1559.681**
0.2	0.3	0.25	0.35	0.1	0.15	0.19	0.23	2	2.5	0.22	0.27	**1538.913**

2.2.2.5 General observations from sensitivity analysis:

It is observed from above table that as Δ_5, Δ_6, Δ_7, Δ_8 increases PF increases and as Δ_1, Δ_2, Δ_3, Δ_4, Δ_9, Δ_{10}, Δ_{11}, Δ_{12} increases PF decreases.

2.3 Conclusion:

From numerical example it can be seen that as profit in case of non-instantaneous deterioration is more than in case of profit in case of instantaneous deterioration.

3.1 Introduction:

An inventory control policy deals with the nature of relative parameters such as deterioration rate, demand, holding cost, shortage cost etc. In classical inventory model it was considered that these parameters are of deterministic and fixed nature but in real life situations these parameters are not deterministic, they are either random or imprecise nature or combination of both. Chapter 2 and 3 deal with inventory models with imprecise nature of parameters. To go more closure to real life situation fuzzy- stochastic environment is more appropriate.

If history related to parameter is known then it is possible to model distribution of that parameter through probability distribution. In this case parameter can be considered as random, for such cases by using probability theory inventory models have been developed. These developed inventory models are known as stochastic inventory models. Karlin (1960) has developed a dynamic inventory model in which demand distributions can change from time to time. He also developed the critical number dependence as a function of stochastic ordering among distributions under different conditions. Kaplan (1970), with stochastic lead time, formed a dynamic inventory model. He developed a probability model for the arrival of outstanding orders in which orders are presumed not to cross in time and the number, size of outstanding orders are independent of the arrival probability. Jing-Sheng Ehrhardt (1984) studied inventory model under the hypothesis that replenishment orders do not cross in time with a stochastic lead time and for a given order, the lead time distribution is independent of the number and size of outstanding orders. Song (1994) analyzed basic continuous time, single item inventory model where demands form a poisson compound process and stochastic lead time. Federgruen and Zipkin(1986) considers a single - item, periodic inventory model with uncertain demands by assuming stationary data and a discrete distribution of demand. Haringa (1999) identifies the best possible reduction in the lead time duration of procurement for stochastic inventory models in conjunction with optimal ordering decisions. The models are established with partial and complete information on the distribution of the lead time demand. Ben-Daya and Haringa (2004) consider stochastic demand with integrated production inventory issue with single buyer single seller and suppose that the lead time varies linearly with lot size. Diwakar *et al.* (2008) constructed a discrete time, randomly demanded retailer time model that proves that the structure optimal policy structure is not affected by credit terms. Maiti (2009) developed an inventory model with probabilistic lead-time, price - dependent demand and shortages. Glock (2012) concentrated on an integrated inventory model for a single client and single purchaser with variable lead time dependent on a lot size and stochastic demand. Darwish *et al.* (2014) constructed inventory model to achieve the best possible lot size and reorder point by taking into account the safety stock corresponds to a safety factor multiplied by the standard deviation of the lead time demand and random demand. For various special cases where demand is presumed to be either exponential or uniform random variables, Zhu *et al.* (2015) studied a periodic review stochastic inventory model. Pal *et al.* (2016) built a production inventory system for decaying items with the manufacturing rate being a random variable and the unit cost of production depends on the size of the lot of production as well as the production rate. Pal *et al.* (2017) formulated stochastic inventory model to achieve minimum average expected cost and best possible production lot size. Mary Dillon (2017) proposed a two - stage stochastic programming model to define optimal periodic review policies for inventory management of red blood cells with a view to minimize operating costs, as well as blood shortage and wastage due to outdating, taking into account perishability and demand uncertainty. Durgam (2017) investigates an alternative way to react to demand, uncertainty in an integrated inventory model, namely the variation of the production rate. Konur (2017) studied an integrated inventory control and scheduling issue with economic and environmental considerations in a stochastic demand. Janssen (2018) developed stochastic multi-item inventory model includes total stock capacity constraints, a positive lead time, a periodic inventory control, a target customer service level for perishable items with a fixed lifetime under a non-stationary random demand. An integrated production-inventory model was formed by Das (2017) for a defective product in the business of a purchaser's and a producer's company. He regarded that beta distribution followed by the defective proportion of the product and further considered that the manufacturer offered the purchaser a defective ratio dependent stochastic credit period to compensate for the losses due to the defective product. Here the lead time follows a normal distribution.

In above discussed model it is assumed that the parameters are random and there probability distributions are known. If parameters are random but if there historical data is not known then randomness of parameter cannot modeled using probability distributions. Such problem is tackled by using fuzzy set theory. The inventory models with fuzzy parameters are known as fuzzy inventory models. Many researchers have been worked in this area. Many researchers studied this research area and developed fuzzy inventory models in various situations. Sommer (1981) utilized Bellman and Zadeh (1970)'s fuzzy dynamic programming to an inventory problem. With fuzzy constraints, Kacprzyk and Staniewski (1982) looked at the best possible company determination from a global view of top management. Park (1987) examines EOQ model from the fuzzy set theoretic perspective. The cost of ordering and the cost of holding are represented by trapezoidal fuzzy numbers. In the planning of material requirements, Lee *et al.* (1990) introduce the use of fuzzy set theory for lot-sizing. Lee and Yao(1996) investigates a group of computing schemas for EOQ as fuzzy values, and the corresponding best possible stock quantity of the inventory with backorder. Petrovika *et al.* (1996) and VujoBevic (1996) given EOQ formula in the presence of imprecise parameters such as inventory costs, such as overage and shortage costs. In a fuzzy environment, Roy and Maiti (1995) solved the classic economic ordered quantity model by considering inventory costs as well as a storage area to be fuzzy.The model is solved through FNLP technique for inventory parameters using different kinds of membership functions. Fuzzy inventory model was formulated by Roy and Maiti (1997) with restricted storage capability and objective functionality as well as the storage area is depicted by a linear membership function. They solved model by FNLP and geometric programming techniques. Roy and Maiti (1998) developed stock dependent demand fuzzy inventory models for decaying items. Total average cost, storage space, inventory costs, purchase and sale prices are intended to be vague. Fuzzy linear membership function used to depict the vagueness of the total average cost and storage area while the triangular fuzzy number represents inventory costs and selling prices. Models were solved by the technique of FNLP. With multiple price breaks Lam and Wong (1996) solved the fuzzy model of Dolan R. J. (1978). They reported that the fuzzy model was more appealing than Dolan (1978)'s crisp model. Yao and Lee (1996, 1998) developed inventory models that considered demand, ordered quantity and quantity of production as vague. Umap (2013) has recently built a fuzzy inventory model for decaying items with holding cost and set up cost are viewed vague. Vagueness is depicted by a hyperbolic membership function. Umap and Bajaj (2014) constructed a fuzzy inventory model for decaying items with demand relying on selling price and advertising frequency in which trapezoidal fuzzy number is used to present parameters. Wasim *et al.* (2016) developed fuzzy inventory system for the decaying item with time-dependent demand. Sahoo *et al.* (2016) constructed a fuzzy inventory model in which rate of decay, demand, the cost of holding, the cost of the unit and salvage value is regarded to be trapezoidal fuzzy numbers. In the cloudy fuzzy environment De *et al.* (2017) deals with the classic backorder economic order quantity inventory model. Shaikh *et al.* (2018) studied a fuzzy inventory model with acceptable delay in payments for a decaying item. Jain *et al.* (2018) obtained fuzzy inventory model with imperfect manufacturing process with all system costs in fuzzy environment. In previous chapter fuzzy inventory models have been discussed for two warehouse system.

The inventory models developed in stochastic as well as in fuzzy environment are discussed above. But to go closer to real life situation combination of stochastic and fuzzy environment known as Fuzzy-stochastic environment is considered by some researchers. In this environment some parameters are considered imprecise and some are random such inventory model is known as Fuzzy-stochastic inventory model. The presence of a blended environment of fuzzy and stochastic in an inventory model is a sensitive phenomenon in real life and the interesting area is the mathematical realization it. Very few fuzzy - stochastic inventory models have been developed. Das *et al.* (2004) developed a multi-item fuzzy-stochastic stock model in which demand and budget resources are deemed to be random and available storage space as well as total expenditure is imprecise. Randomness and impreciseness are expressed by using normal distribution and linear membership function respectively. The Das *et al.* (2004) model was extended by Panda and Kar (2005) by considering price as random variable. Das and Maiti (2011) developed an inventory model for manufacturing process by taking into account one restriction in a fuzzy environment and another in both a fuzzy and stochastic environment. Janna *et al.* (2014) developed an inventory model by presuming the time horizon in a stochastic environment and rate of deterioration as well as budget in a fuzzy environment. Recently Naserabadi (2014) depicted lead time and inflation rate through the triangular membership function and weibull distribution deterioration rate. In present model extension of chance constrained programming to fuzzy environment has been investigated through an inventory model.

In this chapter an inventory model is developed in stochastic, fuzzy and fuzzy-stochastic environments by considering stock dependent demand. In stochastic inventory model purchasing cost and investment goal are expressed as random variable with normal distribution. The stochastic inventory model has been formulated as a stochastic nonlinear programming problem and then reduced to equivalent crisp model using chance constraint programming (CCP) technique. Using FNLP and IFO techniques crisp problem is solved. In fuzzy model holding cost and budget are considered as imprecise. Impreciseness is expressed through linear membership function. In Fuzzy-stochastic model, purchasing cost and investment goal are expressed as random variable with normal distribution and profit as well as available storage space is assumed to be imprecise and vague. Impreciseness is expressed through linear membership function. The fuzzy-stochastic inventory problem is first converted to an equivalent fuzzy problem and then to equivalent crisp problem using linear membership functions. FNLP and IFO techniques are used to solve crisp problem. Using Numerical examples models are illustrated and sensitivity analysis is provided.

3.2 Fuzzy-Stochastic Inventory model for decaying items with stock dependent demand:

Here, inventory model for decaying items with stock dependent demand is developed in stochastic, fuzzy and fuzzy-stochastic environment.

3.2.1 Assumptions:

- Replenishment is instantaneous,
- Lead time is zero
- Selling price is known and constant
- Shortages are not allowed.

3.2.2 Notations:

C_i: Purchasing cost per unit of i^{th} item

P_i: Selling price per unit of i^{th} item

Qi: Initial stock level of i^{th} item

θi: Decaying rate of i^{th} item

Di(t) : Demand rate of per unit of i^{th} item = a_i+bi*Qi(t)

Qi(t) : Inventory level at time t of i^{th} item

C_{1i}: Holding cost per unit of i^{th} item

C_{di}: Decaying cost per unit of i^{th} item

T : Time Period for each cycle of i^{th} item

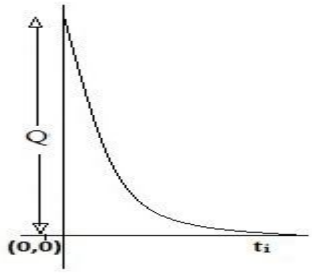
w_i :space required per item of i^{th} item.

W: Available Storage space

B: Total Budget Available for purchasing item

('~' represents the fuzzification of the parameters and '^'represents randomization of parameter)

3.2.3 Figure:



The initial stock level is Q at time t = 0, then inventory level reduces prominently due to demand and partly due to deterioration. At time $t = t_i$, the stock reaches to zero level.

3.2.4 Crisp Model:

Let the differential equation describing the state of inventory in the interval (0, ti) is given by

$$\frac{dQ_i(t)}{dt} + \theta_i * t * Q_i(t) = -(a_i + b_i * Q_i(t)), 0 \le t \le t_i \ -(1)$$

Solving above differential equation using boundary condition $Q_i(0)=Qi$

$$Q_i(t) = \left(-a_i \left[t + \frac{\theta_i * t^3}{6} + \frac{b_i * t^2}{2} \right] + Qi \right) * e^{-\left(\frac{\theta_i * t^2}{2} + b_i * t \right)}, 0 \le t \le t_i \ -(2)$$

The above equation can be simplified using series form of exponential term and ignoring second and higher terms as follows

$$Q_i(t) = \left(-a_i \left[t - \frac{2 * \theta_i * t^3}{6} - \frac{b_i * t^2}{2} \right] + Qi \left[1 - \frac{\theta_i * t^2}{2} - b_i * t \right] \right), 0 \le t \le t_i \ -(2)$$

Using boundary condition $Q_i(t_i)=0$

$$\left(-a_i \left[t_i - \frac{b_i * t_i^2}{2} - \frac{\theta_i * t_i^3}{3} \right] + Qi \left[1 - \frac{\theta_i * t_i^2}{2} - b_i * t_i \right] \right) = 0 - (3)$$

Holding cost over the time period (0 , ti) is given by ,

$$C_{1i} * \int_0^{t_i} Q_i(t)dt = C_{1i} * \left(-a_i \left[\frac{t_i^2}{2} - \frac{b_i * t_i^3}{6} - \frac{\theta_i * t_i^4}{12} \right] + Qi \left[t_i - \frac{\theta_i * t_i^3}{6} - \frac{b_i * t_i^2}{2} \right] \right)$$

Total deterioration cost is given by

$$C_{di} * \int_0^{t_i} t_i * \theta_i * Q_i(t)dt = C_{di} * \theta_i * \left(-a_i \left[\frac{t_i^3}{3} - \frac{b_i * t_i^4}{8} - \frac{\theta_i * t_i^5}{15} \right] + Qi \left[\frac{t_i^2}{2} - \frac{\theta_i * t_i^4}{8} - \frac{b_i * t_i^3}{3} \right] \right)$$

Then the total profit becomes

$$PF = \sum_{i=1}^{n} (P_i - C_i) * Q_i - C_{1i} \int_0^{t_i} Q_i(t)dt + C_{di} * \theta_i \int_0^{t_i} t_i * Q_i(t)dt$$

$$PF = \sum_{i=1}^{n} \left(\begin{array}{l} (P_i - C_i) * Q_i - C_{1i} * \left(-a_i \left[\frac{t_i^2}{2} - \frac{b_i * t_i^3}{6} - \frac{\theta_i * t_i^4}{12} \right] + Qi \left[t_i - \frac{\theta_i * t_i^3}{6} - \frac{b_i * t_i^2}{2} \right] \right) \\ -C_{di} * \theta_i * \left(-a_i \left[\frac{t_i^3}{3} - \frac{b_i * t_i^4}{8} - \frac{\theta_i * t_i^5}{15} \right] + Qi \left[\frac{t_i^2}{2} - \frac{\theta_i * t_i^4}{8} - \frac{b_i * t_i^3}{3} \right] \right) \end{array} \right)$$

Hence the problem is to maximize profit subject to limitations on investment and storage area. That is

$$\begin{array}{l} \max PF \\ \text{Subject to} \\ \sum_{i=1}^{n} w_i * Q_i \le W \\ \sum_{i=1}^{n} C_i * Q_i \le B \\ \left(\begin{array}{l} -a_i \left[t_i - \frac{b_i * t_i^2}{2} - \frac{\theta_i * t_i^3}{3} \right] \\ +Qi \left[1 - \frac{\theta_i * t_i^2}{2} - b_i * t_i \right] \end{array} \right) = 0 \\ Q_i \ge 0, i=1,2,....,n \end{array}$$

3.2.5 Stochastic Model:

In above crisp model parameters are considered to be known and fix. Here it is considered that C_i's and investment are random in nature and there randomness is expressed using normal distribution. Then model in 4.2.3 becomes

$$
\begin{aligned}
&\max \widetilde{PF} \\
&\text{Subject to} \\
&\sum_{i=1}^{n} w_i * Q_i \le W \\
&\sum_{i=1}^{n} C_i * Q_i \le \tilde{B} \\
&\left(-a_i \left[t_i - \frac{b_i * t_i^2}{2} - \frac{\theta_i * t_i^3}{3} \right] + Q_i \left[1 - \frac{\theta_i * t_i^2}{2} - b_i * t_i \right] \right) = 0 \\
&Q_i \ge 0,\ i = 1, 2, \ldots, n
\end{aligned}
$$

The stochastic inventory model has been formulated as a stochastic nonlinear programming problem and then reduced to equivalent crisp model using chance constraint programming (CCP) technique. Then by using the technique of FNLP and IFO crisp problem is solved.

3.2.5.1 Stochastic Non-Linear Programming (SNLP) methodology:

General optimization problem is to find X which minimizes

$$
\begin{aligned}
&\text{Min } k_0(X) \\
&\text{Subject to} \\
&m_j(X) \ge d_j (j = 1, 2, \ldots m) \\
&\text{and } X \ge 0
\end{aligned}
$$

When both objective and constraints are of stochastic nature then above problem become stochastic non-linear programming problem with objective and constraints as:

$$
\begin{aligned}
&\text{Min } k_0(X) \\
&\text{Subject to} \\
&P(k_j \le 0) \ge p_j \ (j = 1, 2, \ldots m) \\
&\text{and } X \ge 0 \\
&\text{where } k_j = m_j(X) - d_j \\
&\text{here } X = (Y_1, Y_2, \ldots, Y_N)^T \text{ is the vector of } N \text{ random variables} \\
&\text{and it includes the decision variables } X_1, X_2, \ldots, X_N
\end{aligned}
$$

The problem stated above can be converted into an equivalent deterministic (crisp) non-linear programming problem using chance constraint programming technique as follows:

The objective function $k_0(X)$ can be expanded about the mean values $\bar{Y}_i$ of random variables Yi as

$$k_0(X) = k_0(\bar{Y}) + \sum_{i=1}^{N} \left(\frac{\delta k_0}{\delta Y_i}\bigg|_{X=\bar{Y}} \right)\left(Y_i - \overline{Y_i}\right) + \text{higher order derivative terms}$$

If the standard deviation σ_{Yi} of random variate Yi are small, $k_0(X)$ can be approximated by the first two terms as

$$k_0(X) \cong k_0(\bar{Y}) + \sum_{i=1}^{N} \left(\frac{\delta k_0}{\delta Y_i}\bigg|_{X=\bar{Y}} \right)\left(Y_i - \overline{Y_i}\right) \equiv \rho(X)$$

If all Yi(i=1,2,…., N) follow normal distribution, then $k_0(X\ )$, which is linear function of X , also follows normal distribution. The mean and variance of ρ(X) are given by

$$\bar{\rho} = \rho(\bar{Y}) = k_0(\bar{Y}) \text{ and } {\sigma_\rho}^2 = \sum_{i=1}^{N} \left(\frac{\delta k_0}{\delta Y_i}\bigg|_{X=\bar{Y}} \right)^2 \sigma_{Y_i}^2$$

Here all Yi's are independent.

As some variables and parameters of the constraints are random in nature, the constraints will be probabilistic and one would like to have the probability of realizing kj<0 must be greater than or equal to specified probability, say, p_j i.e.

$$P(k_j \le 0) \ge p_j$$

$$\text{i.e. } \int_{-\infty}^{0} K_j \, dk_j \ge p_j \text{ -----(*)}$$

Where Kj is the probability density function of the random variable kj(X) whose range is assumed to be −∞ to ∞.The constraint function kj(X) can be expanded around the vector of mean values of the random variables X as

$$k_j \cong k_j(\bar{Y}) + \sum_{i=1}^{N} \left(\frac{\delta k_j}{\delta Y_i}\bigg|_{X=\bar{Y}} \right)\left(Y_i - \overline{Y_i}\right)$$

One can get the mean value and the standard deviation of kj as follows:

$$\bar{k} = k_j(\bar{Y}) \text{ and } {\sigma_{k_j}}^2 = \sum_{i=1}^{N} \left(\frac{\delta k_j}{\delta Y_i}\bigg|_{X=\bar{Y}} \right)^2 \sigma_{Y_i}^2$$

By introducing the new variable

$$\Theta j = \left(\frac{k_j - \overline{k_j}}{\sigma_{k_j}} \right) \sim N(0,1)\ (j = 1,2,.....,n)$$

Equation (*) can be written as

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{\bar{k}_j}{\sigma_{k_j}}} e^{-\frac{\Theta_j^2}{2}}\, d\Theta_j \geq p_j$$

If $\varphi_j(x)$ is the cumulative distribution function of the standard normal distribution evaluated at x and if sj denotes the value of the standard normal variate at which $\varphi_j(sj) = 1 - pj$ then

$$1-p_j = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{s_j} e^{-\frac{t^2}{2}}\, dt$$

$$\text{i.e. } p_j = \frac{1}{\sqrt{2\pi}}\int_{s_j}^{\infty} e^{-\frac{t^2}{2}}\, dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-s_j} e^{-\frac{t^2}{2}}\, dt$$

Therefore

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{\bar{k}_j}{\sigma_{k_j}}} e^{-\frac{\Theta_j^2}{2}}\, d\Theta_j \geq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-s_j} e^{-\frac{t^2}{2}}\, dt$$

$$\text{so } -\frac{\bar{k}_j}{\sigma_{k_j}} = -s_j$$

$$\bar{k}_j - s_j\sigma_{k_j} \leq 0$$

$$\bar{k}_j - s_j\left(\sum_{i=1}^{N}\left(\left.\frac{\delta k_j}{\delta Y_i}\right|_{X=\bar{Y}}\right)^2 \sigma_{Y_i}^2\right)^{1/2} \leq 0$$

Hence, the stochastic programming problem is reduced to multi-objective deterministic non-linear programming problem with objectives and constraints as follows

$$\min \bar{\rho}$$
$$\min \sigma_\rho$$
subject to

$$\bar{k}_j - s_j\left(\sum_{i=1}^{N}\left(\left.\frac{\delta k_j}{\delta Y_i}\right|_{X=\bar{Y}}\right)^2 \sigma_{Y_i}^2\right)^{1/2} \leq 0 \text{-----}(**)$$

Using above methodology model in 4.2.3 becomes crisp multi-objective inventory problem with maximizing expected profit and minimizing variance of profit as follows

$$\text{Max } E_{PF} = \sum_{i=1}^{n}\left(\begin{array}{l}(P_i - \overline{C}i)*Q_i - C_{1i}*\left(-a_i\left[\frac{t_i^2}{2} - \frac{b_i*t_i^3}{6} - \frac{\theta_i*t_i^4}{12}\right] + Qi\left[t_i - \frac{\theta_i*t_i^3}{6} - \frac{b_i*t_i^2}{2}\right]\right) \\ -C_{di}*\theta_i*\left(-a_i\left[\frac{t_i^3}{3} - \frac{b_i*t_i^4}{8} - \frac{\theta_i*t_i^5}{15}\right] + Qi\left[\frac{t_i^2}{2} - \frac{\theta_i*t_i^4}{8} - \frac{b_i*t_i^3}{3}\right]\right)\end{array}\right)$$

$$\text{Min } V_{PF} = \sum_{i=1}^{n}\left(\sigma_{Ci}^2 * Q_i^2\right)$$

Subject to

$$\sum_{i=1}^{n} w_i * Q_i \le W$$

$$\sum_{i=1}^{n} \overline{C}i * Q_i - \overline{B} - 1.96*\left[\sum_{i=1}^{n}\left(\sigma_{Ci}^2 * Q_i^2\right) + \sigma^2_B\right]^{1/2} \le 0$$

$$\left(-a_i\left[t_i - \frac{b_i*t_i^2}{2} - \frac{\theta_i*t_i^3}{3}\right] + Qi\left[1 - \frac{\theta_i*t_i^2}{2} - b_i*t_i\right]\right) = 0$$

To solve above crisp multi-objective non-linear programming problem generally two algorithms are used namely

A) Fuzzy Nonlinear Programming Technique (FNLP) and

B) Intuitionistic Fuzzy Optimization (IFO)

A) Fuzzy Nonlinear Programming Technique (FNLP):

This method is given by Zimmerman (1978). This algorithm has following steps

Step a) Solve the multi goals as a single goal by only considering one goal and neglecting others. Get a solution for each goal.

Step b) Determine the respective values for each goal for each solution acquired in the above step, gives a set of solutions.

Step c) From set of solutions decide lower bound (L_k)and upper bounds (U_k) for k^{th} objective function.

Step d) define the linear membership function $\mu_k(X)$, corresponding to the k^{th} objective

$$\mu_k(Z_k) = \begin{cases} 1 & Z_k < L_k \\ \dfrac{U_k - Z_k}{U_k - L_k} & L_k \le Z_k \le U_k \\ 0 & Z_k > U_k \end{cases}$$

According to Zimmermann, the equivalent crisp non-linear programming problem is

$$\begin{aligned}&\max \alpha\\&\text{subject to}\\&\frac{U_k - Z_k}{U_k - L_k} \geq \alpha,\ k = 1,2\end{aligned}$$

B) Intuitionistic fuzzy optimization (IFO):

This technique is introduced by Angelov (1997). This algorithm has following steps

Step a) Solve the multi goals as a single goal by only considering one goal and neglecting others. Get a solution for each goal.

Step b) Determine the respective values for each goal for each solution acquired in the above step, gives a set of solutions.

Step c) From set of solutions decide lower bound (L_k) and upper bounds (U_k) for k^{th} objective function

Step d) Define the membership function $\mu_k(X)$ as well as non-membership function $v_k(X)$ corresponding to the k^{th} objective lower bounds and upper bounds for non-membership functions are given by L^k and U^k where $U^k=U_k$ and $L^k>L_k$ (Since In case of minimization problem, the lower bound for non-membership function is always greater than that of the membership function that has been proved by S. Banerjee and T. K. Roy (2010))

$$\mu_k(Z_k) = \begin{cases} 1 & Z_k < L_k \\ \dfrac{U_k - Z_k}{U_k - L_k} & L_k \leq Z_k \leq U_k \\ 0 & Z_k > U_k \end{cases} \qquad v_k(Z_k) = \begin{cases} 1 & Z_k < U^k \\ \dfrac{Z_k - L^k}{U^k - L^k} & L^k \leq Z_k \leq U^k \\ 0 & Z_k < L^k \end{cases}$$

Following Angelov together with linear membership function and non-membership functions of an intuitionistic fuzzy optimization model problem can be written as

$$\begin{aligned}&\max \alpha - \beta\\&\text{subject to}\\&\frac{U_k - Z_k}{U_k - Lk} \geq \alpha,\ k = 1,2\\&\frac{Z_k - L^k}{U^k - L^k} \leq \beta,\ k = 1,2\\&\beta \geq 0, \alpha > \beta, \alpha + \beta < 1\end{aligned}$$

By following FNLP, crisp multi-objective non-linear programming problem becomes

$$\max = \alpha$$
Such that
$$\left(\frac{E_{PF} - L_1}{U_1 - L_1}\right) \geq \alpha$$
$$\left(\frac{U_2 - V_{PF})}{U_2 - L_2}\right) \geq \alpha$$
$$\sum_{i=1}^{n} w_i * Q_i \leq W$$
$$\sum_{i=1}^{n} \overline{C}i * Q_i - \overline{B} - 1.96 * \left[\sum_{i=1}^{n}\left(\sigma^2_{Ci} * Q_i^{\ 2}\right) + \sigma^2{}_B\right]^{1/2} \leq 0$$
$$\left(-a_i\left[t_i - \frac{b_i * t_i^{\ 2}}{2} - \frac{\theta_i * t_i^{\ 3}}{3}\right] + Qi\left[1 - \frac{\theta_i * t_i^{\ 2}}{2} - b_i * t_i\right]\right) = 0$$

By following IFO, crisp multi-objective non-linear programming problem becomes

$$\max = \alpha - \beta$$
Such that
$$\left(\frac{E_{PF} - L_1}{U_1 - L_1}\right) \geq \alpha$$
$$\left(\frac{U_2 - V_{PF})}{U_2 - L_2}\right) \geq \alpha$$
$$\left(\frac{U^1 - E_{PF}}{U^1 - L^1}\right) \leq \beta$$
$$\left(\frac{V_{PF} - L^2}{U^2 - L^2}\right) \leq \beta$$
$$\sum_{i=1}^{n} w_i * Q_i \leq W$$
$$\sum_{i=1}^{n} \overline{C}i * Q_i - \overline{B} - 1.96 * \left[\sum_{i=1}^{n}\left(\sigma^2_{Ci} * Q_i^{\ 2}\right) + \sigma^2{}_B\right]^{1/2} \leq 0$$
$$\left(-a_i\left[t_i - \frac{b_i * t_i^{\ 2}}{2} - \frac{\theta_i * t_i^{\ 3}}{3}\right] + Qi\left[1 - \frac{\theta_i * t_i^{\ 2}}{2} - b_i * t_i\right]\right) = 0$$
$$\alpha \geq \beta, \alpha + \beta = 1$$

3.2.6 Fuzzy Model:

Here it is considered that holding cost and budget are imprecise and vague in nature, hence they are represented by fuzzy numbers. Hence model in 4.2.3 converted to fuzzy model as follows

$$\max \tilde{PF} = \sum_{i=1}^{n} \tilde{PF}(Q_i)$$

Subject to

$$\sum_{i=1}^{n} w_i * Q_i \le W$$

$$\sum_{i=1}^{n} C_i * Q_i \le \tilde{B}$$

$$\left(-a_i\left[t_i - \frac{b_i * t_i^2}{2} - \frac{\theta_i * t_i^3}{3}\right] + Qi\left[1 - \frac{\theta_i * t_i^2}{2} - b_i * t_i\right]\right) = 0$$

$$Q_i \ge 0, i = 1, 2, ..., n$$

Let impreciseness of holding cost and budget is expressed using linear membership function then above fuzzy model converted into crisp problem using FNLP as

$$\text{Max} = \alpha$$

Subject to

$$1 - \frac{Co - \tilde{PF}}{P_{\tilde{PF}}} \ge \alpha$$

$$\sum_{i=1}^{n} w_i * Q_i \le W$$

$$1 - \frac{\sum_{i=1}^{n} C_i * Q_i - B}{P_B} \ge \alpha$$

$$\left(-a_i\left[t_i - \frac{b_i * t_i^2}{2} - \frac{\theta_i * t_i^3}{3}\right] + Qi\left[1 - \frac{\theta_i * t_i^2}{2} - b_i * t_i\right]\right) = 0$$

$$Q_i \ge 0, i = 1, 2, ..., n$$

$$\text{where } \tilde{PF} = \sum_{i=1}^{n} \left(\begin{array}{l} (P_i - C_i) * Q_i - \left(C_{1i} + (1-\beta) * P_{C_{1i}}\right) * \left(-a_i\left[\frac{t_i^2}{2} - \frac{b_i * t_i^3}{6} - \frac{\theta_i * t_i^4}{12}\right] + Qi\left[t_i - \frac{\theta_i * t_i^3}{6} - \frac{b_i * t_i^2}{2}\right]\right) \\ -C_{di} * \theta_i * \left(\begin{array}{l} -a_i\left[\frac{t_i^3}{3} - \frac{b_i * t_i^4}{8} - \frac{\theta_i * t_i^5}{15}\right] \\ +Qi\left[\frac{t_i^2}{2} - \frac{\theta_i * t_i^4}{8} - \frac{b_i * t_i^3}{3}\right] \end{array} \right) \end{array} \right)$$

3.2.7 Fuzzy-Stochastic Model:

Here, C_i's and investment are considered to be random and there randomness is expressed using normal distribution. Hence Storage area becomes fuzzy and its fuzziness represented by linear membership function, then model 4.2.3 converted into fuzzy-stochastic model as follows

$$\max PF = \sum_{i=1}^{n} PF(Q_i)$$

Subject to

$$\sum_{i=1}^{n} w_i * Q_i \le \tilde{W}$$

$$\sum_{i=1}^{n} \tilde{C}i * Q_i \le \tilde{B}$$

$$Q_i \ge 0,\ i = 1, 2, ..., n$$

Using SNLP the above model becomes

$$\text{Max } E_{PF} = \sum_{i=1}^{n} \left(\begin{array}{l} (P_i - \bar{C}i) * Q_i - C_{1i} * \left(-a_i \left[\frac{t_i^2}{2} - \frac{b_i * t_i^3}{6} - \frac{\theta_i * t_i^4}{12} \right] + Qi \left[t_i - \frac{\theta_i * t_i^3}{6} - \frac{b_i * t_i^2}{2} \right] \right) \\ -C_{di} * \theta_i * \left(\begin{array}{l} -a_i \left[\frac{t_i^3}{3} - \frac{b_i * t_i^4}{8} - \frac{\theta_i * t_i^5}{15} \right] \\ +Qi \left[\frac{t_i^2}{2} - \frac{\theta_i * t_i^4}{8} - \frac{b_i * t_i^3}{3} \right] \end{array} \right) \end{array} \right)$$

$$\text{Min } V_{PF} = \sum_{i=1}^{n} \left(\sigma_{Ci}^2 * Q_i^2 \right)$$

Subject to

$$1 - \frac{\sum_{i=1}^{n} w_i * Q_i - W}{P_W} \ge \alpha$$

$$\sum_{i=1}^{n} \bar{C}i * Q_i - \bar{B} - 1.96 * \left[\sum_{i=1}^{n} \left(\sigma_{Ci}^2 * Q_i^2 \right) + \sigma_B^2 \right]^{1/2} \le 0$$

$$\left(-a_i \left[t_i - \frac{b_i * t_i^2}{2} - \frac{\theta_i * t_i^3}{3} \right] + Qi \left[1 - \frac{\theta_i * t_i^2}{2} - b_i * t_i \right] \right) = 0$$

The above problem is converted to crisp problem by using FNLP method as follows

$$\max\ \alpha$$

Such that

$$\left(\frac{E_{PF} - L_1}{U_1 - L_1}\right) \geq \alpha$$

$$\left(\frac{U_2 - V_{PF})}{U_2 - L_2}\right) \geq \alpha$$

$$1 - \frac{\sum_{i=1}^{n} w_i * Q_i - W}{P_W} \geq \alpha$$

$$\sum_{i=1}^{n} \bar{C}i * Q_i - \bar{B} - 1.96 * \left[\sum_{i=1}^{n}\left(\sigma_{Ci}^2 * Q_i^{\,2}\right) + \sigma^2{}_B\right]^{1/2} \leq 0$$

$$\left(-a_i\left[t_i - \frac{b_i * t_i^{\,2}}{2} - \frac{\theta_i * t_i^{\,3}}{3}\right] + Qi\left[1 - \frac{\theta_i * t_i^{\,2}}{2} - b_i * t_i\right]\right) = 0$$

Similarly using IFO method crisp problem obtained as follows

$$\max \alpha - \beta$$

Such that

$$\left(\frac{E_{PF} - L_1}{U_1 - L_1}\right) \geq \alpha$$

$$\left(\frac{U_2 - V_{PF})}{U_2 - L_2}\right) \geq \alpha$$

$$\left(\frac{U^1 - E_{PF}}{U^1 - L^1}\right) \leq \beta$$

$$\left(\frac{V_{PF} - L^2}{U^2 - L^2}\right) \leq \beta$$

$$1 - \frac{\sum_{i=1}^{n} w_i * Q_i - W}{P_W} \geq \alpha$$

$$\sum_{i=1}^{n} \bar{C}i * Q_i - \bar{B} - 1.96 * \left[\sum_{i=1}^{n}\left(\sigma_{Ci}^2 * Q_i^{\,2}\right) + \sigma^2{}_B\right]^{1/2} \leq 0$$

$$\left(-a_i\left[t_i - \frac{b_i * t_i^{\,2}}{2} - \frac{\theta_i * t_i^{\,3}}{3}\right] + Qi\left[1 - \frac{\theta_i * t_i^{\,2}}{2} - b_i * t_i\right]\right) = 0$$

$$\alpha \geq \beta, \alpha + \beta = 1$$

3.2.8 Numerical Example:

a) Crisp Model

Input:

C_1=7, C_2=6.75 Rs. , P_1= P_2=10 Rs., C_{11}= 2Rs.,C $_{12}$=2.2 Rs., a_1=100, a_2=110, b_1=b_2=0.5, B=850 Rs. , W=275 Sq. ft, w_1=2 Sq. ft, w_2=2.2 Sq. ft, θ_1=0.05, θ_2=0.06, C_{d1}= C_{d2}=5 Rs., T=1yr.

Output:

Q_1=48.23, Q_2=74.86, PF=323.54 Rs., t_1= 0.44 yr., t_2= 0.59 yr.

b) Stochastic Model

Input :

$\hat{C}_1 \sim N(7, 0.01)$, $\hat{C}_2 \sim N(6.75, 0.015)$, $\hat{B} \sim N(850,100)$, P_1= P_2=10 Rs., C_{11}=2,C_{12}=2.2 Rs., a_1=100, a_2=110, b_1=b_2=0.5, B=850 Rs., W=275 Sq. ft, w_1=2Sq. ft, w_2=2.2Sq. ft, θ_1=0.05, θ_2=0.06, C_{d1}= C_{d2}=5Sq. ft, Co =340 Rs., $P_{E_{PF}}$ =50, Do =15, $P_{V_{PF}}$ =5,T=1yr.

Output:

1. FNLP method:

α=0.89, E_{PF} =334.77Rs. , V_{PF} =10.52Rs. ,Q_1=54.91,Q_2=73.28,t_1=0.47yr. , t_2=0.55yr.

2. IFO method:

α=0.83,β=0.16, E_{PF} =331.54Rs. , V_{PF} =10.0 1 Rs. ,Q_1=66.20,Q_2=61.34,t_1=0.47yr. , t_2=0.55r.

c) Fuzzy Model

Input:

C_1=7, C2=6.75Rs. , P_1= P_2=10Rs. , C_{11}= 2Rs.,C $_{12}$=2.2Rs., a_1=100, a_2=110, b_1=b_2=0.5, , W=275 Sq. ft, w_1=2 Sq. ft, w_2=2.2 Sq. ft, θ_1=0.05, θ2=0.06, C_{d1}= C_{d2}=5Rs. , T=1yr.,B=850Rs. ,P_B=130

Output:

FNLP method

α=0.88,Q_1=50.58, Q_2=75.74, PF=330.86Rs., t_1= 0.45 yr., t_2= 0.59 yr.

d) Fuzzy- Stochastic Model

Input:

$\hat{C}_1 \sim N(7, 0.01)$, $\hat{C}_2 \sim N(6.75, 0.015)$, $\hat{B} \sim N(850, 100)$, P_1= P_2=10 Rs., C_{11}= C_{12}=2.2 Rs., a_1=100, a_2=110, b_1=b_2=0.5, B=850 Rs., W=275 Sq. ft, w_1=2Sq. ft, w_2=2.2Sq. ft, θ_1=0.05, θ2=0.06, C_{d1}= C_{d2}=7Sq. ft, Co =337.94, $P_{E_{PF}}$ =40, Do =10.3093, $P_{V_{PF}}$ =2, P_W =20,T=1yr.

Output:

1. FNLP method:

α=0.86, E_{PF} =335.17Rs. , V_{PF} =10.72Rs. ,Q_1=51.73,Q_2=76.67,t_1=0.45yr. , t_2=0.58yr.

2. IFO method:

α=0.75,β=0.25, E_{PF} =335.17Rs. , V_{PF} =10.72Rs. ,Q_1=54.92,Q_2=73.29,t_1=0.47yr. , t_2=0.56yr.

3.2.9: Sensitivity analysis

- **Stochastic Model**

Table 3.1: Effect of Change of values in mean of C_1

Average value of C_1	α	β	Q_1	t_1	Q_2	t_2	E_{PF}	V_{PF}
5	0.96		89.17	0.70	40.05	0.33	502.21	10.18
	0.91	0.09	83.33	0.66	37.47	0.31	472.92	9.95
6.5	0.88		87.59	0.69	45.37	0.37	378.33	10.37
	0.88	0.12	75.27	0.61	50.12	0.40	360.50	10.46
7	0.80		54.92	0.47	73.29	0.56	334.77	10.52
	0.75	0.25	54.92	0.47	73.29	0.56	334.77	10.52

Table 3.2: Effect of Change of values in var of C_1

Variation of C_1	α	β	Q_1	t_1	Q_2	t_2	E_{PF}	V_{PF}
0.01	0.90		54.92	0.47	73.29	0.56	334.77	10.52
	0.75	0.25	54.92	0.47	73.29	0.56	334.77	10.52
0.013	0.73		72.14	0.59	54.09	0.43	326.37	10.56
	0.72	0.28	62.52	0.53	65.25	0.51	332.92	10.71
0.016	0.64		65.71	0.55	57.73	0.46	322.07	10.91
	0.65	0.35	58.89	0.50	66.97	0.52	329.20	11.08

Table 3.3: Effect of Change of values in Mean of C_2

Average value of C_2	α	β	Q_1	t_1	Q_2	t_2	E_{PF}	V_{PF}
6	0.95		65.31	0.55	64.43	0.50	385.14	10.24
	0.91	0.09	61.08	0.52	59.95	0.47	362.65	10.62
6.5	0.93		63.47	0.53	66.77	0.51	355.07	10.35
	0.86	0.14	67.78	0.56	58.11	0.46	341.72	10.81
6.75	0.90		54.92	0.47	73.29	0.56	334.77	10.52
	0.75	0.25	54.92	0.47	73.29	0.56	334.77	10.52

Table 3.4: Effect of Change of values in var of C_2

Variation of C_2	α	β	Q_1	t_1	Q_2	t_2	E_{PF}	V_{PF}
0.015	0.90		54.92	0.47	73.29	0.56	334.77	10.52
	0.75	0.25	54.92	0.47	73.29	0.56	334.77	10.52
0.018	0.86		63.54	0.53	64.30	0.50	332.86	10.71
	0.71	0.29	63.54	0.53	64.30	0.50	332.86	10.71
0.021	0.81		69.23	0.57	58.36	0.46	330.71	10.93
	0.68	0.32	69.23	0.57	58.36	0.46	330.71	10.93

Table 3.5: Effect of Change of values in mean of Budget

Average value of Budget	α	β	Q_1	t_1	Q_2	t_2	E_{PF}	V_{PF}
830	0.78		46.54	0.41	79.11	0.59	328.84	10.75
	0.65	0.35	46.54	0.41	79.11	0.59	328.84	12.37
850	0.90		54.92	0.47	73.29	0.56	334.77	10.52
	0.75	0.25	54.92	0.47	73.29	0.56	334.77	10.52
870	0.95		68.31	0.57	62.16	0.49	337.72	10.23
	0.80	0.20	68.31	0.57	62.16	0.49	337.72	11.30

Table 3.6: Effect of Change of values in Var of Budget

Variation of budget	α	β	Q_1	t_1	Q_2	t_2	E_{PF}	V_{PF}
25	0.87		51.35	0.45	76.25	0.57	333.43	10.66
	0.72	0.28	51.35	0.45	76.25	0.57	333.43	12.18
50	0.88		52.61	0.46	75.20	0.57	333.93	10.61
	0.73	0.27	52.61	0.46	75.20	0.57	333.93	12.10
100	0.90		54.92	0.47	73.29	0.56	334.77	10.52
	0.75	0.25	54.92	0.47	73.29	0.56	334.77	10.52

Fuzzy Model:

Table 3.7: Effect of Change of values in P_B

P_B	α	Q_1	t_1	Q_2	t_2	PF
70	0.82	50.53	0.45	75.39	0.59	329.96
90	0.85	50.56	0.45	75.54	0.60	330.35
110	0.87	50.57	0.45	75.66	0.60	330.64
130	0.88	50.58	0.45	75.75	0.60	330.86

Table 3.8: Effect of Change of values in B

B	α	Q_1	t_1	Q_2	t_2	PF
775	0.46	50.10	0.45	73.28	0.58	324.38
800	0.60	50.29	0.45	74.09	0.59	326.59
825	0.74	50.45	0.45	74.92	0.59	328.76
850	0.88	50.58	0.45	75.75	0.60	330.86

Fuzzy-Stochastic Model

Table 3.9: Effect of Change of values in mean of C_1

Average value of C_1	α	β	Q_1	t_1	Q_2	t_2	E_{PF}	V_{PF}
5	0.96		75.89	0.62	55.65	0.44	489.24	10.20
	0.82	0.18	75.10	0.61	55.07	0.44	484.75	10.09
6.5	0.96		75.89	0.62	55.65	0.44	375.42	10.20
	0.82	0.18	75.10	0.61	55.07	0.44	372.10	10.09
7	0.86		51.73	0.45	76.67	0.58	335.17	10.72
	0.75	0.25	54.92	0.47	73.29	0.56	334.77	10.52

Table 3.10: Effect of Change of values in var of C_1

Variation of C_1	α	β	Q_1	t_1	Q_2	t_2	E_{PF}	V_{PF}
0.01	0.86		51.73	0.45	76.67	0.58	335.17	10.72
	0.75	0.25	54.92	0.47	73.29	0.56	334.77	10.52
0.013	0.81		55.25	0.48	73.06	0.55	334.98	10.94
	0.71	0.29	61.49	0.52	66.46	0.51	333.53	10.74
0.016	0.74		58.91	0.50	69.12	0.53	334.09	11.28
	0.64	0.36	58.43	0.50	68.56	0.53	331.83	11.19

Table 3.11: Effect of Change of values in Mean of C_2

Average value of C_2	α	β	Q_1	t_1	Q_2	t_2	E_{PF}	V_{PF}
6	0.96		75.89	0.62	55.65	0.44	379.21	10.20
	0.82	0.18	75.10	0.61	55.07	0.44	375.85	10.09
6.5	0.93		63.91	0.54	66.29	0.51	354.76	10.33
	0.81	0.19	69.65	0.58	59.98	0.47	350.35	10.12
6.75	0.86		51.73	0.45	76.67	0.58	335.17	10.72
	0.75	0.25	54.92	0.47	73.29	0.56	334.77	10.52

Table 3.12: Effect of Change of values in var of C_2

Variation of C_2	α	β	Q_1	t_1	Q_2	t_2	E_{PF}	V_{PF}
0.015	0.86		51.73	0.45	76.67	0.58	335.17	10.72
	0.75	0.25	54.92	0.47	73.29	0.56	334.77	10.52
0.018	0.78		54.38	0.49	70.84	0.54	334.65	11.10
	0.70	0.30	62.09	0.52	65.83	0.51	333.36	10.80
0.021	0.72		61.92	0.52	66.14	0.51	333.71	11.41
	0.65	0.35	66.07	0.55	61.73	0.48	332.15	11.12

Table 3.13: Effect of Change of values in mean of Budget

Average value of Budget	α	β	Q_1	t_1	Q_2	t_2	E_{PF}	V_{PF}
830	0.73		39.03	0.35	87.05	0.64	328.20	11.35
	0.65	0.35	44.09	0.39	81.70	0.61	328.77	10.93
850	0.86		51.73	0.45	76.67	0.58	335.17	10.72
	0.75	0.25	54.92	0.47	73.29	0.56	334.77	10.52
870	0.94		66.74	0.56	63.82	0.50	338.36	10.28
	0.80	0.20	68.31	0.57	62.16	0.49	337.72	10.23

Table 3.14: Effect of Change of values in Var of Budget

Variation of budget	α	β	Q_1	t_1	Q_2	t_2	E_{PF}	V_{PF}
25	0.83		48.21	0.42	79.58	0.59	333.62	10.87
	0.72	0.28	51.35	0.45	76.25	0.57	333.43	10.66
50	0.84		49.44	0.43	78.57	0.59	334.19	10.82
	0.73	0.27	52.61	0.46	75.20	0.57	333.93	10.61
100	0.86		51.73	0.45	76.67	0.58	335.17	10.72
	0.75	0.25	54.92	0.47	73.29	0.56	334.77	10.52

3.2.10 Conclusion:

From table 3.7 and 3.8 it is observed that as goal and aspiration level of budget increases level of satisfaction α and PF increases. Table 3.1,3.3, 3.9 and 3.11, it is seen that as average of purchase cost increases level of satisfaction α, average value of PF decreases and variance of PF increase. Table 3.2, 3.4, 3.10 and 3.12 shows that as variance of purchase cost increases, level of satisfaction α, average value of PF decreases and variance of PF increases. Table 3.5, 3.6, 3.13 and 3.14 shows that as average and variance of budget increases, level of satisfaction α, average value of PF increases and variance of PF decreases. Especially from numerical example it is noted that Fuzzy-stochastic model gives more accurate profit values than stochastic, fuzzy and crisp models.

Chapter 4 Multi-objective Multi-Item Fuzzy and

4.1: Introduction

Decision making is an important part of daily life. It ranges from individual to largest groups. Decision making problems includes simple as well as complex problem. In previous chapters inventory models are developed for single objective but in some situations decision maker came across multiple objectives. Complex problems may be multi-objective problems. Multi-objective problem includes several statements such as maximizing some objectives and minimizing remaining objectives. Multi-objective decision making problem address decision makers desired state of system. Three levels of goal setting used in multi-objective decision problem, these levels are a) maximizing all objectives b) minimizing all objectives c) minimizing some objectives and maximizing others.Multi-objective decision making is used in various fields. Haimes *et al.* (1979) considered a planning model of water and land resources for Maumee River Basin. Candler and Norton (1977) used multilevel programming in their technical report. Nagata (1995) explored production and transportation problems of multiple products and factories as multi-objective problem.

Inventory in real life problem meets more than one goal. These goals can be to maximize profit, minimize the cost of waste and minimize the cost of shortages. To accomplish them concurrently, multi-objective inventory problem engaged above objectivesimultaneously. Such problems are modeled as decision-making problems with various objectives. Most of the time these goals contradict one another. In general, from a set of feasible alternatives, the decision maker (DM) chooses a compromise solution.Studying multi-objective inventory models is therefore essential. Till now many researchers studied these models in crisp and imprecise environment.

Regarding multi-objective crisp inventory, Mandal and maiti (1998) developed multi-item, multi-objective inventory model under space and budget restriction with stock dependent demand. They considered selling price as well as holding cost dependent on purchasing price. Model is solved using FNLP technique. Mahapatra and Maiti (2005) proposed multi-item, multi-objective inventory model for manufacturing process with stock dependent demand. The model aimed at maximizing profit and minimizing the overall cost of manufacturing.Under restricted warehouse space and budget restriction, the model is invented. Using Fuzzy additive programming and FNLP, the model is solved. Mahapatra *et al.* (2005) formulated multi-objective and single-objective, inventory models for stochastically decaying items, where demand is stock dependant and the selling price of commodities.For each item, profit maximization goals are developed individually with separate goals and the goal programming method used to achieve compromise solutions to the multi-objective inventory problem.They used different kinds of membership functions to represent goals, i.e. linear, quadratic, exponential, etc. Kar and Roy (2008) developed multi-objective inventory model for manufacturing process in which production rate is function of unit cost. Objectives of paper were to maximize profit and minimize production cost as well as wastage cost and solution was obtained by using FNLP and Fuzzy goal programming technique. Wee and Lo (2009) constructed multi-objective inventory model by using fuzzy demand with shortages. An objective of paper was to maximize profit as well as return on inventory investment. Solution was obtained by using inverse weight FNLP and fuzzy additive goal programming method. Prasah and Seshaiah (2011) built multi-objective inventory model without shortages and with budget and warehouse space constraint. Objectives of paper were to minimize total expenditure of organization and reducing number of warehouse allocations. Model is illustrated numerically using FNLP.

Often objectives, parameters of inventory model are imprecise. To tackle impreciseness fuzzy set theory is used. Faritha *et al.* (2010) built a multi-item, multi-objective fuzzy inventory model to maximize profit and minimize the cost of waste under budget and available space restrictions. They converted fuzzy inventory model to crisp model using fuzzy ranking method and crisp model is solved using FNLP. Chakrabortty *et al.* (2011) developed multi item multi-objective inventory model with exponential demand and considered carrying cost, shortage cost in fuzzy environment. Model is solved using IFO. Dash *et al.* (2013) constructed fuzzy multi item multi-objective inventory model. They converted fuzzy inventory model to crisp model using ranking method. Crisp model is solved using FNLP. Chakrabortty *et al.* (2013) developed an inventory model for manufacturing process with demand, carrying cost and shortage cost as vague. Model is solved using FNLP. Multi-objective inventory model based on stock dependent demand is constructed by Jadhav and Bajaj (2013) in fuzzy as well as crisp environment. Under the restricted budget and space constraint, the model is formed. The model aims to minimize the average total cost as well as the cost of wastage and maximize profit. Solution of model is obtained using different techniques such as FNLP, Weighted Fuzzy programming Technique, Weighted goal programming. Gholamian *et al.* (2015) developed supply chain model under uncertainty of demand and solved by multi-objective optimization method. The model is demonstrated using a case study. Kumar and Debashish (2015) constructed fuzzy multi-item, multi-objective inventory model under four constraints. The model is solved by using multi-objective fuzzy goal programming. Asma *et al.* (2016) proposed fuzzy multi-item, multi-objective inventory model based on stock dependent demand. They transformed the fuzzy inventory model to a crisp model using the fuzzy ranking technique and crisp model is solved using IFO. Khalifehzadeh *et al.* (2017) studied multi-objective production distribution system in presence of fuzzy parameters. The model is formulated and solved as a mixed integer programming model using the Ranking Genetic Algorithm (RGA) and Concessive Variable Neighborhood Search (CVNS). Garai *et al.* (2019) proposed fuzzy multi-objective multi-item inventory model.

If some parameters are fuzzy and some are stochastic then multi-objective inventory problem is solved in mixed environment known as multi-objective Fuzzy-stochastic inventory model. Till now multi-item, multi-objective inventory models have not been developed in Fuzzy-stochastic environment.

In this chapter two inventory models have been developed. The first inventory model is derived in fuzzy environment for deteriorating items with exponential demand. In this model it is considered that holding cost, deterioration cost and shortage cost are imprecise and there impreciseness is expressed using linear membership function. The second inventory model is constructed in fuzzy-stochastic environment for deteriorating items with demand function is dependent on price and advertisement. Here it is assumed that purchasing cost, shortage cost as well as budget are considered to be random in nature and there randomness is represented by normal distribution while storage space is considered to be imprecise and its impreciseness is expressed through linear membership function. By applying Stochastic non-linear programming fuzzy stochastic inventory model is converted into multi-objective crisp model. The crisp versions of both models have been solved by using FNLP and IFO technique.

4.2 Model I: Multi-objective Multi-item fuzzy inventory model with Exponential demand

Here inventory model is developed for decaying items with exponential demand in fuzzy environment. This model is gives compromise solution for two objectives. These two objectives are maximizing profit and minimizing shortage cost subject to budget and warehouse constraints.

4.2.1 Assumptions:

- Replenishment is instantaneous.
- Shortages are allowed.
- Selling price is known and constant.

4.2.2 Notations:

C_i : Purchasing cost per unit of ith item

P_i : Selling price per unit ofith item

Q_i Initial stock level of ith item

θ_i : Deterioration rate of ith item

$D(t) = a_i e^{b_i t_i}$: Demand rate of per unit of ith item

$Q_i(t)$: Inventory level at time t of ith item

C_{1i} : Holding cost per unit ofith item

C_{di} : Decaying cost per unit of ith item

C_{2i} : Shortage cost per unit of ith item

w_i : Warehouse space required for ith item

T : Time period for each cycle

W : Warehouse space available to store items

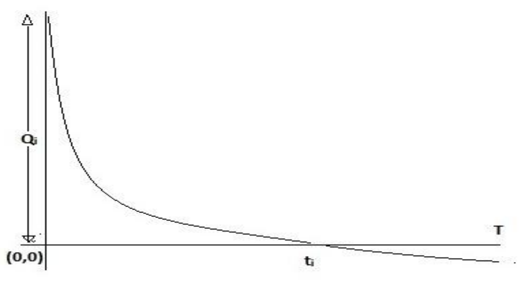
B: Total Budget Available for purchasing item

PF: Total Profit

SC: Total Shortage Cost

('~' represents the fuzzification of the parameters and '^'represents randomization of parameter)

4.2.3 Figure:



The initial stock level is Q at time t = 0, then inventory level reduces prominently due to demand and partly due to deterioration. At time t = $t_{i,}$ the stock reaches to zero level, then shortages occurs and accumulate up to time t=T.

4.2.4 Crisp model:

Let the differential equation describing state of inventory in time period $(0,t_i)$ is as follows

$$\frac{dQ_i(t)}{dt} + \theta_i Q_i(t) = -a_i e^{b_i t}, 0 \le t \le t_i \qquad (1)$$

Solution of above differential equation using boundary condition

$Q_i(0) = Q_i$

$$Q_i(t) = -a_i \left[\frac{e^{b_i t}}{b_i + \theta_i} \right] + \left(Q_i + \frac{a_i}{b_i + \theta_i} \right) e^{-\theta_i t}, 0 \le t \le t_i \qquad (2)$$

Using boundary condition $Q_i(t_i) = 0$ in above equation

$$-a_i \left[\frac{e^{b_i t_i}}{b_i + \theta_i} \right] + \left(Q_i + \frac{a_i}{b_i + \theta_i} \right) e^{-\theta_i t_i} = 0$$

using series form of exponential term and ignoring secondand higher terms

$$\Rightarrow t_i = \left[\frac{1}{\theta_i + b_i} \right] \ln \left(1 + \frac{Q_i(b_i + \theta_i)}{a_i} \right)$$

The differential equation describing the state of inventory in time period (t_i, T) is given by following equation

$$\frac{dQ_i(t)}{dt} = -a_i e^{b_i t}, t_i \le t \le T_i \qquad (3)$$

Integrating both sides and solving using condition $Q_i(t_i) = 0$

$$Q_i(t) = -\left(\frac{a_i e^{b_i t}}{b_i} \right) + \left(\frac{a_i e^{b_i t_i}}{b_i} \right), t_i \le t \le T_i$$

Total Holding cost over the time period $(0,t_i)$ is given by following equation

$$C_{1i}\int_0^{t_i} Q_i(t)dt = C_{1i}\left(\left(\frac{a_i}{b_i*(b_i+\theta_i)}\right)\left(1-e^{b_i t_i}\right)+\left(Q_i+\frac{a_i}{(b_i+\theta)_i}\right)\left(\frac{1}{\theta_i}\right)\left(1-e^{\theta_i t_i}\right)\right).$$

Total shortage cost is given by following equation

$$C_{2i}\left(-\int_{t_i}^{T_i} Q_i(t)dt\right) = C_{2i}\left(-\frac{a_i}{b_i^2}\left(e^{b_i t_i}-e^{b_i T_i}\right)-\frac{a_i}{b_i}\left(T_i-t_i\right)e^{b_i t_i}\right)$$

Then the total profitis given by following equation

$$PF = \sum_{i=1}^{n}\left((P_i - C_i)Q_i - (C_{1i}+C_{di}\theta)\int_0^{t_i} Q_i(t)dt\right).$$

$$PF = \sum_{i=1}^{n}\left((P_i - C_i)Q_i - (C_{1i}+C_{di}\theta)\left(\begin{array}{l}\left(\frac{a_i}{b_i(b_i+\theta)_i}\right)\left(1-e^{b_i t_i}\right)\\ +\left(Q_i+\frac{a_i}{(b_i+\theta)_i}\right)\left(\frac{1}{\theta_i}\right)\left(1-e^{\theta_i t_i}\right)\end{array}\right)\right).$$

Then the total shortage cost is given by following equation

$$SC = \sum_{i=1}^{n}\left(C_{2i}\left(-\frac{a_i}{b_i^2}\left(e^{b_i t_i}-e^{b_i T_i}\right)-\frac{a_i}{b_i}\left(T_i-t_i\right)e^{b_i t_i}\right)\right)$$

Hence the inventory problem is maximizing profit as well as minimizing shortage cost subject to limitations on budget and storage area is given by

MaxPF

Min SC

Subject to

$$\sum_{i=1}^{n} w_i Q_i \le W$$

$$\sum_{i=1}^{n} C_i Q_i \le B \qquad \text{----------(4)}$$

$$t_i = \left[\frac{1}{\theta_i + b_i}\right]\ln\left(1+\frac{Q_i(b_i+\theta_i)}{a_i}\right)$$

$$Q_i \ge 0,\ i = 1, 2, ..., n$$

4.2.5 Fuzzy model:

In above crisp model, holding cost, deterioration cost and shortage cost are assumed to be imprecise and there impreciseness is expressed using linear membership function.

Budget is also considered to be imprecise and its impreciseness expressed using quadratic membership function then above crisp

$$\text{Max}\ \tilde{PF}$$
$$\text{Min}\ \tilde{SC}$$
Subject to
$$\sum_{i=1}^{n} w_i Q_i \le W$$
$$\sum_{i=1}^{n} C_i Q_i \le \tilde{B}$$
$$t_i = \left[\frac{1}{\theta_i + b_i}\right] \ln\left(1 + \frac{Q_i (b_i + \theta_i)}{a_i}\right) \qquad --------(5)$$
$$Q_i \ge 0,\ i = 1, 2, ..., n$$

$$\text{where}\ \tilde{PF} = \sum_{i=1}^{n}\left((P_i - C_i)Q_i - (\tilde{C}_{1i} + \tilde{C}_{di}\theta)\left(\left(\frac{a_i}{b_i(b_i+\theta)_i}\right)\left(1 - e^{b_i t_i}\right) + \left(Q_i + \frac{a_i}{(b_i+\theta)_i}\right)\left(\frac{1}{\theta_i}\right)\left(1 - e^{\theta_i t_i}\right)\right)\right)$$

$$\tilde{SC} = \sum_{i=1}^{n}\left(\tilde{C}_{2i}\left(-\frac{a_i}{b_i^{\,2}}\left(e^{b_i t_i} - e^{b_i T_i}\right) - \frac{a_i}{b_i}\left(T_i - t_i\right)e^{b_i t_i}\right)\right)$$

Using FNLP method above model converted into crisp model as follows

$$\text{Max}\ \alpha$$
Such that
$$\left(\frac{U_1 - SC}{U_1 - L_1}\right) \ge \alpha$$
$$\left(\frac{PF - L_2}{U_2 - L_2}\right) \ge \alpha$$
$$\sum_{i=1}^{n} w_i * Q_i \le W$$
$$\left(\frac{B - \sum_{i=1}^{n} C_i * Q_i}{P_B}\right)^2 \ge \alpha$$
$$Q_i \ge 0,\ i = 1, 2, ..., n$$

$$t_i = \left[\frac{1}{\theta_i + b_i}\right] * \ln\left(1 + \frac{Q_i * (b_i + \theta_i)}{a_i}\right)$$

$$PF = \sum_{i=1}^{n}\left((P_i - C_i)Q_i - \left(\left[C_{1i} - (1-\alpha)p_{C_{1i}}\right] + \left[C_{di} - (1-\alpha)p_{C_{di}}\right]\theta\right)\left(\begin{array}{l}\left(\frac{a_i}{b_i(b_i+\theta)_i}\right)\left(1 - e^{b_i t_i}\right) \\ + \left(Q_i + \frac{a_i}{(b_i+\theta)_i}\right)\left(\frac{1}{\theta_i}\right)\left(1 - e^{\theta_i t_i}\right)\end{array}\right)\right)$$

$$SC = \sum_{i=1}^{n}\left(\left(C_{2i} - (1-\alpha)p_{C_{2i}}\right)\left(-\frac{a_i}{b_i^{2}}\left(e^{b_i t_i} - e^{b_i T_i}\right) - \frac{a_i}{b_i}\left(T_i - t_i\right)e^{b_i t_i}\right)\right)$$

(Where α is satisfaction level) in step d) of FNLP method discussed in chapter IV under point 4.2.5.1, exponential membership function is used instead of linear membership function, then above crisp model is converted into following model

Max α

Such that

$$\frac{e^{-v\left(\frac{U_1-SC}{U_1-L_1}\right)}-e^{-v}}{1-e^{-v}} \geq \alpha$$

$$\frac{e^{-w\left(\frac{PF-L_2}{U_2-L_2}\right)}-e^{-v}}{1-e^{-v}} \geq \alpha$$

$$\sum_{i=1}^{n} w_i * Q_i \leq W$$

$$\left(\frac{B-\sum_{i=1}^{n} C_i * Q_i}{P_B}\right)^2 \geq \alpha$$

$$Q_i \geq 0, i = 1, 2, ..., n$$

$$t_i = \left[\frac{1}{\theta_i + b_i}\right] * \ln\left(1 + \frac{Q_i * (b_i + \theta_i)}{a_i}\right)$$

$$PF = \sum_{i=1}^{n}\left((P_i - C_i)Q_i - \left(\left[C_{1i} - (1-\alpha)p_{C_{1i}}\right] + \left[C_{di} - (1-\alpha)p_{C_{di}}\right]\theta\right)\left(\begin{array}{l}\left(\frac{a_i}{b_i(b_i+\theta)_i}\right)\left(1-e^{b_i t_i}\right) \\ +\left(Q_i + \frac{a_i}{(b_i+\theta)_i}\right)\left(\frac{1}{\theta_i}\right)\left(1-e^{\theta_i t_i}\right)\end{array}\right)\right)$$

$$SC = \sum_{i=1}^{n}\left(\left(C_{2i} - (1-\alpha)p_{C_{2i}}\right)\left(-\frac{a_i}{b_i^2}\left(e^{b_i t_i} - e^{b_i T_i}\right) - \frac{a_i}{b_i}\left(T_i - t_i\right)e^{b_i t_i}\right)\right)$$

(Where α is satisfaction level)

In IFO method discussed in chapter IV under point 4.2.5.1, membership function is linear membership function and nonlinear membership function is quadratic membership function then fuzzy model is converted to following model

$$\text{Max } \alpha - \beta$$

Subject to

$$\frac{U_r - Z_r}{U_r - L_r} \geq \alpha, \; (r = 1, 2)$$

$$\left(\frac{Z_r - L^r}{U^r - L^r} \right)^2 \leq \beta, \; (r = 1, 2)$$

$$\sum_{i=1}^{n} w_i Q_i \leq W$$

$$\left(\frac{B - \sum_{i=1}^{n} C_i * Q_i}{P_B} \right)^2 \geq \alpha$$

$$t_i = \left[\frac{1}{\theta_i + b_i} \right] * \ln \left(1 + \frac{Q_i * (b_i + \theta_i)}{a_i} \right)$$

$$PF = \sum_{i=1}^{n} \left((P_i - C_i) Q_i - \left(\left[C_{1i} - (1-\alpha) p_{C_{1i}} \right] + \left[C_{di} - (1-\alpha) p_{C_{di}} \right] \theta \right) \left(\begin{array}{l} \left(\frac{a_i}{b_i (b_i + \theta)_i} \right) \left(1 - e^{b_i t_i} \right) + \\ \left(Q_i + \frac{a_i}{(b_i + \theta)_i} \right) \left(\frac{1}{\theta_i} \right) \left(1 - e^{\theta_i t_i} \right) \end{array} \right) \right)$$

$$SC = \sum_{i=1}^{n} \left(\left(C_{2i} - (1-\alpha) p_{C_{2i}} \right) \left(-\frac{a_i}{b_i^2} \left(e^{b_i t_i} - e^{b_i T_i} \right) - \frac{a_i}{b_i} \left(T_i - t_i \right) e^{b_i t_i} \right) \right)$$

$$Q_i \geq 0, \; i = 1, 2, ..$$

$$\beta \geq 0, \alpha > \beta, \alpha + \beta < 1$$

(Where α is satisfaction level and β is non-satisfaction level)

In above crisp model exponential membership function is used instead of linear membership function then above crisp model converted to following model

Max $\alpha - \beta$

Subject to

$$\frac{e^{-v\left(\frac{U_1 - SC}{U_1 - L_1}\right)} - e^{-v}}{1 - e^{-w}} \ge \alpha$$

$$\frac{e^{-v\left(\frac{PF - L_2}{U_2 - L_2}\right)} - e^{-v}}{1 - e^{-w}} \ge \alpha$$

$$\left(\frac{U_1 - SC}{U_1 - L_1}\right) \le \beta$$

$$\left(\frac{PF - L_2}{U_2 - L_2}\right) \le \beta$$

$$\sum_{i=1}^{n} w_i Q_i \le W$$

$$\left(\frac{B - \sum_{i=1}^{n} C_i * Q_i}{P_B}\right)^2 \ge \alpha$$

$$t_i = \left[\frac{1}{\theta_i + b_i}\right] * \ln\left(1 + \frac{Q_i * (b_i + \theta_i)}{a_i}\right)$$

$$PF = \sum_{i=1}^{n}\left((P_i - C_i)Q_i - \left(\left[C_{1i} - (1-\alpha)p_{C_{1i}}\right] + \left[C_{di} - (1-\alpha)p_{C_{di}}\right]\theta\right)\left(\left(\frac{a_i}{b_i(b_i+\theta)_i}\right)\left(1 - e^{b_i t_i}\right) + \left(Q_i + \frac{a_i}{(b_i+\theta)_i}\right)\left(\frac{1}{\theta_i}\right)\left(1 - e^{\theta_i t_i}\right)\right)\right)$$

$$SC = \sum_{i=1}^{n}\left(\left(C_{2i} - (1-\alpha)p_{C_{2i}}\right)\left(-\frac{a_i}{b_i^2}\left(e^{b_i t_i} - e^{b_i T_i}\right) - \frac{a_i}{b_i}\left(T_i - t_i\right)e^{b_i t_i}\right)\right)$$

$Q_i \ge 0, i = 1, 2, ..$

$\beta \ge 0, \alpha > \beta, \alpha + \beta < 1$

(Where α is satisfaction level and β is non-satisfaction level)

4.2.6 Numerical example:

a) Crisp Model:

Input:

$P_1 = P_2 = 10Rs., C_1 = 7Rs., C_2 = 6.75Rs., C_{11} = C_{12} = 2.2Rs., C_{d1} = C_{d2} = 4Rs., \theta_1 = 0.05, \theta_1 = 0.06, a_1 = 100,$
$a_2 = 50, b_1 = 0.25, b_2 = 0.5 w_1 = 2Sq.ft, w_2 = 2.2Sq.ft, w = 200Sq.ft, B = 650Rs., C_{21} = C_{22} = 1Rs.$

Output:

Using LINGO Software, following results are obtained.

1) By FNLP method:

$$\alpha = 0.7618, Q_1 = 58.15965, Q_2 = 35.98258, t_1 = 0.5360 yr., t_2 = 0.604672 yr.,$$
$$PF = 225.6571 Rs., SC = 18.4467 Rs..$$

2) By IFO method:

$$\alpha = 0.76183, \beta = 0.17957, Q_1 = 58.15965, Q_2 = 35.98258, t_1 = 0.536082 yr., t_2 = 0.604672 yr.,$$
$$PF = 225.6571 Rs., SC = 18.4467 Rs.$$

b) Fuzzy Model:

Input:

$$P_1 = P_2 = 10 Rs., C_1 = 7 Rs., C_2 = 6.75 Rs., C_{11} = C_{12} = 2.2 Rs., C_{d1} = C_{d2} = 4 Rs., \theta_1 = 0.05, \theta_1 = 0.06, a_1 = 100$$
$$a_2 = 50, b_1 = 0.25, b_2 = 0.5 w_1 = 2 Sq.ft, w_2 = 2.2 Sq.ft, w = 200 Sq.ft, B = 650 Rs, C_{21} = C_{22} = 1 Rs.$$
$$p_{C_{h1}} = p_{C_{h2}} = p_{C_{d1}} = p_{C_{d2}} = p_{C_{21}} = p_{C_{22}} = 0.01, p_B = 50, v = 0.1.$$

Output:

Using LINGO Software, following results are obtained.

1) By FNLP method with Linear membership function

$$\alpha = 0.7030, Q_1 = 58.1643, Q_2 = 40.0147, t_1 = 0.5361 yr., t_2 = 0.6612 yr.,$$
$$PF = 232.6576 Rs., SC = 17.0203 Rs..$$

2) By FNLP method with Exponential membership function

$$\alpha = 0.7372, Q_1 = 58.0180, Q_2 = 39.9265, t_1 = 0.5349 yr., t_2 = 0.6600 yr.,$$
$$PF = 229.9214 Rs., SC = 17.1153 Rs..$$

3) By IFO method with Linear membership function and Quadratic nonlinear membership function

$$\alpha = 0.3680, \beta = 0.1354, Q_1 = 52.5787, Q_2 = 47.6589, t_1 = 0.4882 yr., t_2 = 0.604672 yr.,$$
$$PF = 234.5057 Rs., SC = 17.5737 Rs..$$

4) By IFO method with exponential membership function and Quadratic nonlinearmembership function

$$\alpha = 0.8678, \beta = 0.0160, Q_1 = 67.7395, Q_2 = 27.6004, t_1 = 0.6167 yr., t_2 = 0.4810 yr.,$$
$$PF = 225.0254 Rs., SC = 18.2089 Rs.$$

4.2.7: Sensitivity analysis

Table 4.1: Effect of Change of values in capital investment

Capital Investment		α	β	Q_1	t_1	Q_2	t_2	SC	PF
630	FNLP with exponential membership function	0.571		56.841	0.525	39.238	0.651	17.883	229.044
650	FNLP with exponential membership function	0.737		58.018	0.535	39.927	0.660	17.115	229.921
670	FNLP with exponential membership function	0.997		63.594	0.582	33.709	0.572	17.020	230.588
690	FNLP with exponential membership function	0.998		65.044	0.594	32.347	0.552	17.017	230.973
630	FNLP with linear membership function	0.610		56.703	0.524	39.155	0.649	17.974	228.663
650	FNLP with linear membership function	0.703		58.164	0.536	40.015	0.661	17.020	232.658
670	FNLP with linear membership function	0.782		59.628	0.549	40.879	0.673	16.092	236.599
690	FNLP with linear membership function	0.814		64.596	0.590	37.090	0.620	15.301	238.359
630	IFO with exponential and quadratic membership function	0.767	0.050	65.431	0.597	29.057	0.503	18.371	224.423
650	IFO with exponential and quadratic membership function	0.868	0.016	67.740	0.617	27.600	0.481	18.209	225.025
670	IFO with exponential and quadratic membership function	0.891	0.011	67.792	0.617	27.635	0.482	18.172	225.179
690	IFO with exponential and quadratic membership function	0.907	0.008	67.828	0.617	27.660	0.482	18.146	225.286
630	IFO with linear and quadratic membership function	0.286	0.082	54.996	0.509	42.560	0.696	17.709	231.216
650	IFO with linear and quadratic membership function	0.368	0.135	52.579	0.488	47.659	0.764	17.574	234.506
670	IFO with linear and quadratic membership function	0.442	0.196	51.230	0.477	51.665	0.815	17.452	237.482
690	IFO with linear and quadratic membership function	0.467	0.218	50.893	0.474	52.937	0.831	17.411	238.492

Table 4.2: Effect of Change of values in available warehouse space

Warehouse space		α	β	Q_1	t_1	Q_2	t_2	SC	PF
180	FNLP with exponential membership function	0.745		62.481	0.573	30.751	0.529	18.764	223.061
200	FNLP with exponential membership function	0.737		58.018	0.535	39.927	0.660	17.115	229.921
230	FNLP with exponential membership function	0.655		58.357	0.538	40.131	0.663	16.896	233.184
250	FNLP with exponential membership function	0.486		58.939	0.543	40.484	0.668	16.522	234.771
180	FNLP with linear membership function	0.872		61.393	0.564	30.077	0.518	19.532	219.917
200	FNLP with linear membership function	0.703		58.164	0.536	40.015	0.661	17.020	232.658
230	FNLP with linear membership function	0.504		58.883	0.542	40.450	0.667	16.557	234.620
250	FNLP with linear membership function	0.254		59.600	0.548	40.888	0.673	16.102	236.565
180	IFO with exponential and quadratic membership function	0.647	0.027	63.146	0.578	31.164	0.535	18.303	224.966
200	IFO with exponential and quadratic membership function	0.868	0.016	67.740	0.617	27.600	0.481	18.209	225.025
230	IFO with exponential and quadratic membership function	0.891	0.011	67.792	0.617	27.635	0.482	18.172	225.179
250	IFO with exponential and quadratic membership function	0.907	0.008	67.828	0.617	27.660	0.482	18.146	225.286
180	IFO with linear and quadratic membership function	0.257	0.066	56.227	0.520	40.495	0.668	17.756	230.067
200	IFO with linear and quadratic membership function	0.368	0.135	52.579	0.488	47.659	0.764	17.574	234.506
230	IFO with linear and quadratic membership function	0.299	0.089	51.913	0.483	48.664	0.777	17.688	234.695
250	IFO with linear and quadratic membership function	0.251	0.063	51.493	0.479	49.307	0.785	17.766	234.806

4.2.8 General observations from sensitivity analysis:

From Tables 4.1 and 4.2, it is observed that capital investment and available warehouse space increases the parameters such as PF, t_1, t_2, Q_1, Q_2 increases and SC decreases. So, to reduce shortage cost and maximize profit decision maker has to increase capital investment and available warehouse space.Satisfaction levelα and non-satisfaction levelβ of decision maker changes with change in capital investment. It is also observed that IFO with membership function represented by linear and non-membership function by quadratic membership function maximizes profit as compared to IFO with membership function represented by exponential and non-membership function by quadratic membership function. FNLP with linear membership maximizes profit as compared to exponential membership function. FNLP works better than IFO in case of minimizing shortage cost.

4.3 Model II: Multi-objective Multi-item Fuzzy-stochastic inventory model with Price and advertisement dependent demand

Here inventory model is developed for decaying items with price and advertisement dependent demand in Fuzzy-stochastic environment. This model is gives compromise solution for two objectives. These two objectives are maximizing profit and minimizing shortage cost subject to budget and warehouse constraints.

4.3.1 Assumptions:

- The scheduling period is constant and no lead-time.
- Replenishment rate is infinite.
- Selling price is known and constant.
- Shortages are allowed

4.3.2 Notations:

C_i: Purchasing cost per unit of i^{th} item

P_i: Selling price per unit of i^{th} item

S_i :Initial stock level of i^{th} item

θ_i: Deterioration rate of i^{th} item

$D= (a_i-b_iP_i)\ N^{\alpha}$: Demand rate of per unit of i^{th} item

$Q_i(t)$: Inventory level at time t of i^{th} item

C_{1i}: Holding cost per unit ofi^{th} item

C_{di}: Decaying cost per unit of i^{th} item

C_{2i}: Shortage cost per unit of i^{th} item

w_i: Warehouse space required for i^{th} item

T:Time period for each cycle

B : Capital investment available for purchasing all items

W: Warehouse space available to store items

PF: Total Profit

SC: Total Shortage cost

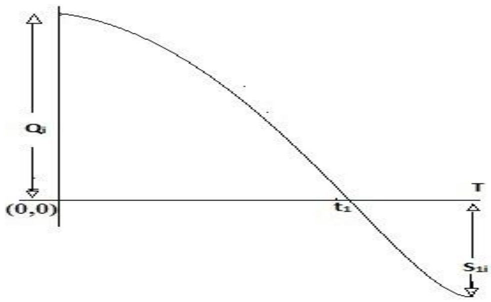
EPF: Total Expected Profit

VPF: Variability in Expected Profit

ESC: Total Expected Shortage cost

VSC: Variability in Expected Shortage cost

4.3.3 Figure:



The initial stock level is Q at time t = 0, then inventory level reduces prominently due to demand and partly due to deterioration. At time $t = t_{i,}$ the stock reaches to zero level,then shortages occur and accumulate to the level S_{1i} at t=T.

4.3.4 Crisp Model:

Let the differential equation describing the state of inventory in the interval $(0, t_1)$ is given by,

$$\frac{dQ_i(t)}{dt} + \theta_i Q_i(t) = -(a_i - b_i p_i)N^{\alpha} \qquad 0 \le t \le t_1 \qquad (1)$$

Solving above differential equation using boundary condition, $Q_i(0) = Q_i$

$$Q_i(t) = -\frac{(a_i - b_i p_i)N^{\alpha}}{\theta_i} + \left[\frac{Q_i\theta_i + (a_i - b_i p_i)N^{\alpha}}{\theta_i}\right]e^{-\theta_i t} \qquad 0 \le t \le t_1 \quad (2)$$

Using boundary condition in above equation, $Q(t_1) = 0$

$$t_1 = \frac{1}{\theta_i}\log\left\{1+\frac{Q_i\theta_i}{(a_i - b_ip_i)N^{\alpha}}\right\}. \qquad (3)$$

And the differential equation describing the state of inventory in the interval (t_1, T) is given by,

$$\frac{dQ_i(t)}{dt} = -(a_i - b_ip_i)N^{\alpha} \quad t_1 \le t \le T \qquad (4)$$

Integrating above equation on both sides and solving using condition, $Q(t_1) = 0$

$$Q_i(t) = -(a_i - b_ip_i)N^{\alpha}t + (a_i - b_ip_i)N^{\alpha}t_1 \qquad t_1 \le t \le T \qquad (5)$$

Using boundary condition in above equation $Q(T) = S_{1i}$

$$S_{1i} = (a_i - b_ip_i)N^{\alpha}T - (a_i - b_ip_i)N^{\alpha}\frac{1}{\theta_i}\log\left\{1+\frac{S_i\theta_i}{(a_i - b_ip_i)N^{\alpha}}\right\} (6)$$

Total decaying cost is obtained as follows

$$C_D = C_{di}\int_0^{t_1}\theta_iQ_i(t)dt \; C_D = C_{di}\theta_i\left[-\frac{(a_i - b_ip_i)N^{\alpha}t_1}{\theta_i} - \left(\frac{S_i\theta_i + (a_i - b_ip_i)N^{\alpha}}{\theta_i^2}\right)\left(e^{-\theta_it_1} - 1\right)\right]$$

(7) The total holding cost is as follows

$$C_H = C_{1i}\int_0^{t_1}Q_i(t)dt$$

$$C_H = C_{1i}\left[-\frac{(a_i - b_ip_i)N^{\alpha}t_1}{\theta_i} - \left(\frac{S_i\theta_i + (a_i - b_ip_i)N^{\alpha}}{\theta_i^2}\right)\left(e^{-\theta_it_1} - 1\right)\right] \qquad (8)$$

Total shortage cost is as follows

$$C_p = C_{2i}\left[-\int_{t_1}^{T}Q_i(t)dt\right]$$

$$C_p = C_{2i}\left[\frac{(a_i - b_ip_i)N^{\alpha}}{2}\left(T - t_1\right)^2\right] \qquad (9)$$

and total advertisement cost is $C_A = C_{ai}(S_i - S_{di})P_iN$

Hence the total profit for i^{th} item becomes

$$PF_i = (P_i - C_i) * S_i - C_H - C_D - C_P - C_A$$

$$PF_i = (P_i - C_i) - \left(C_{1i} + C_{di}\theta_i\right)\left[-\frac{(a_i - b_i p_i)N^{\alpha} t_1}{\theta_i} - \left(\frac{Q_i\theta_i + (a_i - b_i p_i)N^{\alpha}}{\theta_i^2}\right)\left(e^{-\theta_i t_1} - 1\right)\right] \tag{10}$$

$$- C_{ai}\left(S_i - \frac{Q_i^2\theta_i}{(a_i - b_i p_i)N^{\alpha}}\right) p_i N$$

The above equation can be simplified using series form of exponential term and ignoring second and higher terms as follows

$$PF_i = (P_i - C_i) - \left(C_{1i} + C_{di}\theta_i\right)\frac{Q_i^2}{(a_i - b_i p_i)N^{\alpha}} - C_{ai}\left(Q_i - \frac{Q_i^2\theta_i}{(a_i - b_i p_i)N^{\alpha}}\right) P_i N$$

The total shortage cost for i[th] item becomes

$$SC_i = C_{2i}\left[\frac{(a_i - b_i p_i)N^{\alpha}}{2}\left(T - \frac{Q_i}{(a_i - b_i p_i)N^{\alpha}}\right)^2\right] \tag{11}$$

Hence the problem is to maximize profit and minimize shortage cost subject to limitations on investment and storage area

$$\text{Max}\, PF = \sum_{i=1}^{n} PF_i$$

$$\text{Min}\, SC = \sum_{i=1}^{n} SC_i$$

subject to

$$\sum_{i=1}^{n} w_i Q_i \le W$$

$$\sum_{i=1}^{n} C_i Q_i \le B$$

$$Q_i \ge 0, \quad i = 1, 2, 3, ---n.$$

4.3.5 Fuzzy-stochastic inventory model:

In above crisp model, purchasing cost, shortage cost as well as budget is considered to be random in nature and there randomness is represented by normal distribution while storage space is considered to be imprecise and its impreciseness is expressed through linear membership function then above crisp model becomes

$$
\left.\begin{array}{l}
\text{Max } \tilde{P}F \\
\text{Min } \tilde{S}C \\
\text{Subject to} \\
\sum\limits_{i=1}^{n} w_i Q_i \le \tilde{W} \\
\sum\limits_{i=1}^{n} C_i S_i \le \tilde{B} \\
S_i \ge 0,\ i = 1, 2, \ldots., n
\end{array}\right\} \text{-----}(12)
$$

where

$$
\tilde{P}F\,i = (P_i - \tilde{C}i) - \left(C_{1i} + C_{di}\theta_i\right)\frac{S_i^{2}}{(a_i - b_i p_i)N^{\alpha}}
$$

$$
- C_{2i}\left[\frac{(a_i - b_i p_i)N^{\alpha}}{2}\left(T - \frac{S_i}{(a_i - b_i p_i)N^{\alpha}}\right)^{2}\right]
$$

$$
- C_{ai}\left(S_i - \frac{S_i^{2}\theta_i}{(a_i - b_i p_i)N^{\alpha}}\right)P_i * N
$$

$$
\tilde{S}C = \sum_{i=1}^{n}\left(\tilde{C}_{2i}\left[\frac{(a_i - b_i p_i)N^{\alpha}}{2}\left(T - \frac{S_i}{(a_i - b_i p_i)N^{\alpha}}\right)^{2}\right]\right)
$$

Using SNLP the above model becomes

$$
\left.\begin{array}{l}
\text{Max EPF} \\
\text{Min VPF} \\
\text{Min ESC} \\
\text{Min VSC} \\
\text{Subject to} \\
\sum\limits_{i=1}^{n} w_i Q_i \le \tilde{W} \\
\sum\limits_{i=1}^{n} \bar{C}i * S_i - \bar{B} - 1.96 * \left[\sum\limits_{i=1}^{n}\left(\sigma_{ci}^{2} * Q_i^{2}\right) + \sigma_{B}^{2}\right]^{1/2} \le 0 \\
S_i \ge 0,\ i = 1, 2, \ldots., n
\end{array}\right\} \text{-----}(13)
$$

where

$$EPF = \sum_{i=1}^{n}\left\{\begin{array}{l}(P_i - \bar{C}i) - \left(C_{1i} + C_{di}\theta_i\right)\dfrac{S_i^{\,2}}{(a_i - b_i p_i)N^{\alpha}} - C_{2i}\left[\dfrac{(a_i - b_i p_i)N^{\alpha}}{2}\left(T - \dfrac{S_i}{(a_i - b_i p_i)N^{\alpha}}\right)^2\right] \\ -C_{ai}\left(S_i - \dfrac{S_i^{\,2}\theta_i}{(a_i - b_i p_i)N^{\alpha}}\right)P_i.N\end{array}\right\}$$

$$VPF = \left(\sum_{i=1}^{n}\left(\sigma_{ci}^2 * S_i^{\,2}\right)\right)^{1/2}$$

$$ESC = \sum_{i=1}^{n}\left(\bar{c}2i\left[\frac{(a_i - b_i p_i)N^{\alpha}}{2}\left(T - \frac{S_i}{(a_i - b_i p_i)N^{\alpha}}\right)^2\right]\right)$$

$$VSC = \left(\sum_{i=1}^{n}\left(\sigma_{c_{2i}}^2\left[\frac{(a_i - b_i p_i)N^{\alpha}}{2}\left(T - \frac{S_i}{(a_i - b_i p_i)N^{\alpha}}\right)^2\right]\right)\right)^{1/2}$$

The above problem is converted to crisp problem by using FNLP method as follows

$\text{Max}\ \alpha$

Such that

$$\left(\frac{EPF - L_1}{U_1 - L_1}\right) \geq \alpha$$

$$\left(\frac{U_2 - VPF}{U_2 - L_2}\right) \geq \alpha$$

$$\left(\frac{U_3 - ESC}{U_3 - L_3}\right) \geq \alpha$$

$$\left(\frac{U_4 - VSC}{U_4 - L_4}\right) \geq \alpha$$

$$\sum_{i=1}^{n}\bar{C}i * Q_i - \bar{B} - 1.96 * \left[\sum_{i=1}^{n}\left(\sigma_{ci}^2 * Q_i^{\,2}\right) + \sigma^2_{B}\right]^{1/2} \leq 0$$

$$\left(\frac{W - \sum_{i=1}^{n} w_i * Q_i}{P_B}\right) \geq \alpha$$

$$S_i \geq 0,\ i = 1, 2, ..., n$$

Similarly using IFO method crisp problem obtained as follows

Max $\alpha - \beta$

Subject to

$$\left(\frac{EPF - L_1}{U_1 - L_1}\right) \geq \alpha$$

$$\left(\frac{U_2 - VPF}{U_2 - L_2}\right) \geq \alpha$$

$$\left(\frac{U_3 - ESC}{U_3 - L_3}\right) \geq \alpha$$

$$\left(\frac{U_4 - VSC}{U_4 - L_4}\right) \geq \alpha$$

$$\left(\frac{U^1 - EPF}{U^1 - L^1}\right) \leq \beta$$

$$\left(\frac{VPF - L^2}{U^2 - L^2}\right) \leq \beta$$

$$\left(\frac{ESC - L^3}{U^3 - L^3}\right) \leq \beta$$

$$\left(\frac{VSC - L^4}{U^4 - L^4}\right) \leq \beta$$

$$\sum_{i=1}^{n} \bar{C}i * Q_i - \bar{B} - 1.96 * \left[\sum_{i=1}^{n}\left(\sigma_{ci}^2 * Q_i^{\ 2}\right) + \sigma^2_B\right]^{1/2} \leq 0$$

$$\left(\frac{W - \sum_{i=1}^{n} w_i * Q_i}{P_B}\right) \geq \alpha$$

$$Q_i \geq 0,\ i = 1, 2, \ldots., n$$

$$\beta \geq 0, \alpha > \beta, \alpha + \beta < 1$$

4.3.6 Numerical example:

a) Crisp Model

Input:

$P_1 = P_2 = 10\,Rs, C_1 = 4Rs, C_2 = 5Rs, C_{11} = 0.7Rs, C_{12} = 0.8Rs, C_{d1} = 4Rs., C_{d2} = 5Rs., \theta_1 = \theta_2 = 0.03, a_1 = 100,$
$a_2 = 90, b_1 = b_2 = 0.5, w_1 = 2Sq.ft, w_2 = 1.2Sq.ft, w = 150Sq.ft, N = 2, \alpha = 0.5, T = 1yr, B = 450Rs, C_{21} = C_{22} = 1Rs..$

Output:

1) By FNLP method

$\alpha = 0.5817, S_1 = 22.0419, S_2 = 72.3664, PF = 314.41Rs., SC = 158.40Rs..$

2) By IFO method

$\alpha = 0.4759, \beta = 0.4759, S_1 = 17.72, S_2 = 75.82, PF = 302.87 Rs., SC = 147.88 Rs..$ **b) Fuzzy-stochastic Model**

Input:

$$P_1 = P_2 = 10Rs., C_{11} = 0.7Rs., C_{12} = 0.8Rs., C_{d1} = 4Rs., C_{d2} = 5Rs., \theta_1 = \theta_2 = 0.03, a_1 = 100,$$
$$a_2 = 90, b_1 = b_2 = 0.5, w_1 = 2Sq.ft, w_2 = 1.2Sq.ft, w = 150Sq.ft, N = 2, \alpha = 0.5, T = 1yr., C_{21} = C_{22} = 1Rs..$$
$$C_1 \sim N(4Rs., 0.01Rs.), C_2 \sim N(5Rs., 0.015Rs.), B \sim (450Rs., 50Rs.), p_w = 20Sq.ft.$$

Output:

1) By FNLP method

$$\alpha = 0.3195, S_1 = 54.9963, S_2 = 21.9985, EPF = 288.75Rs, VPF = 6.12Rs, ESC = 67.04Rs, VSC = 13.40Rs.$$

2) By IFO method

$$\alpha = 0.3076, \beta = 0.3076, S_1 = 49.60, S_2 = 30.78, EPF = 298.77Rs., VPF = 6.23Rs., ESC = 63.48Rs., VSC = 12.69Rs.$$

4.7 Sensitivity analysis:

Table 4.3 Effect of change of values in capital investment:

Capital Investment		α	β	Q_1	Q_2	ESC	EPF	VSC	VPF
280	FNLP	0.31		59.50	14.35	70.93	277.82	14.19	6.20
300	FNLP	0.32		55.36	21.40	67.32	287.97	13.46	6.12
330	FNLP	0.32		55.00	22.00	67.04	288.75	13.41	6.12
280	IFO	0.29	0.34	62.63	8.85	74.16	268.73	14.83	6.36
300	IFO	0.32	0.32	58.22	16.56	69.73	281.18	13.95	6.16
330	IFO	0.31	0.31	50.89	28.73	64.23	296.68	12.85	6.19

Table 4.4: Effect of change of values in available warehouse space

Warehouse space		α	β	Q_1	Q_2	ESC	EPF	VSC	VPF
150	FNLP	0.32		55.00	22.00	67.04	288.75	13.41	6.12
180	FNLP	0.18		66.00	26.40	57.30	363.36	11.46	7.35
200	FNLP	0.09		73.33	29.33	51.38	412.15	10.28	8.17
150	IFO	0.31	0.31	49.61	30.78	63.49	298.78	12.70	6.23
180	IFO	0.18	0.59	66.00	26.40	57.30	363.36	11.46	7.35
200	IFO	0.09	0.79	73.33	29.33	51.38	412.15	10.28	8.17

4.3.8 General observations from sensitivity analysis:

From Tables 4.3 and 4.4, it is observed that as capital investment and available warehouse space increases PF increases and SC decreases. So,to reduce shortage cost and maximize profit decision maker has to increase capital investment and available warehouse space. Satisfaction level α and non-satisfaction level β of decision maker changes with change in capital investment. From numerical example it can be seen that Fuzzy-stochastic inventory model gives accurate profit as well as shortage cost output as compared to crisp model. Fuzzy-stochastic inventory models output gives expected value of i.e. on an average value that can be achieved for profit and shortage cost with their respective variations. Crisp model gives one value of profit but Fuzzy-stochastic model gives range of variation of profit with average value, so this model gives more insight when some parameters are stochastic and some are fuzzy. IFO works better than FNLP in case of minimizing shortage cost.

4.4 Chapter Conclusion:

Generally it can be seen that fuzzy and Fuzzy-stochastic inventory model gives more accurate solution as compared to crisp inventory model. These models are more realistic than crisp inventory model. These models handle randomness and uncertainties so study of these models is important. These techniques are appropriate to tackle inventory models in real life situations.

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INVENTORY MODELS WITH UNCERTAIN DATA

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ABOUT AUTHORS

Dr. Rahul Waliv is working as Assistant Professor in the Department of Statistics at Kisan Veer Mahvidyalaya, Wai (M.S.), India affiliated to Shivaji University, Kolhapur. He has completed M. Sc. in Statistics and obtained Ph.D. in the field of Operations Research. He has 11 years experience of teaching to UG classes. He published 5 research papers in national and international journals. He also attended and presented his research work in many national and international level conferences, workshops, seminars, etc. Apart from the academic duties he is working on various college level committees.

Dr. Hemant Umap is working as Head of the department of Statistics at Yashavantrao Chavan Institute of Science, Satara an Autonomous college affiliated to Shivaji University, Kolhapur. He has completed M.Sc. in Statistics and obtained Ph.D. in the field of Operations Research. He has 26 years experience of teaching and to UG and PG classes. He published 17 research papers in national and international journals. He has published two text books and two reference books in Statistics. One student awarded Ph.D. degree and four are working for Ph.D. under his guidance. He has worked as Resource person and chairperson in many national and international conferences. He is member of Board of Studies (BoS) of Statistics in Shivaji University, Kolhapur and Chairman of BoS of Statistics of Yashavantrao Chavan Institute of Science, Satara (an Autonomous college).

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