Journal of Science Research International (JSRI)
ISSN: 2456-6365 (Online)

OPEN ACCESS

A STUDY OF INVOLUTORY ADDITION CAYLEY GRAPHS USING TOPOLOGICAL INDICES

E. Lavanya[1], Ch. Meena Kumari[1] and M. Siva Parvathi[2]

[1]*Department of BS&H, Avanthi Institute of Engineering and Technology, Thagarapuvalasa, Vizianagaram, Andhra Pradesh, India.*

[2]*Department of Applied Mathematics, Sri Padmavati Mahila Visvavidyalayam, Tirupati, Andhra Pradesh, India*

**Corresponding author E-mail: eegalagautam@gmail.com, chinthalameenakumari@gmail.com, parvathimani2008@gmail.com*

Received: 17 June 2026	*Revised: 17 July 2026*	*Accepted: 12 August 2026*	*Published: 30 August 2026*

DOI: https://doi.org/10.5281/zenodo.22316284

Abstract:

Discrete Mathematics is a part of mathematics which manages comprehension of discrete structures of issues in day-to-day life. Graph theory is one of the most advanced branches of discrete mathematics with variety of applications to different branches of Science and Technology. Topological indices play a crucial role in graph theory, especially in quantitative structure-property relationship (QSPR) and quantitative structure-activity relationship (QSAR). Topological index is a measurement based on the structure of the graph, by taking Involutory Addition Cayley Graphs $G^+(Z_n, I_v)$ the author's evaluated graph related topological indices. In this paper, topological index values like the Zero-connectivity index, first connectivity index and second connectivity index of the Involutory addition cayley graphs $G^+(Z_n, I_v)$ were discussed.

Keywords: *Involutory Addition Cayley Graphs, Zero Connectivity Index, First Connectivity Index, Second Connectivity Index.*

AMS Subject Classification: *05C40, 05C19.*

1. Introduction

A graph $G(V, E)$is a mathematical object that may be thought of as a collection of edges and a set of vertices that connect any or all of the vertices. In a graph G, two vertices are considered neighboring if an edge joins them; otherwise, the edge is considered non-adjacent. We indicate that a graph G has $V(G)$vertices and E(G) edges, accordingly. The cardinality of $V(G)$is the definition of the order of G. The cardinality of $E(G)$is represented by $|E|$, and that of $V(G)$ by |V |. The number of edges that occur with a vertex v in a graph G is known as its degree, or deg(v). For a positive integer $n > 1$, the involutory addition Cayley graph $G^+(Z_n, I_v)$, is the graph whose vertex set is $Z_n = \{0,1,2,3, \dots, n-1\}$ and edge set $E(G_n) = \{xy \ : x, y \,\epsilon\, Z_n, x + y \,\epsilon\, I_v\}$, where $I_v = \{x \,\epsilon\, Z_n : x^2 \equiv 1 (mod\ n)\}$ is the set of involutory elements of Z_n.

A topological index can be simply described as a graph-related mathematical number that is invariant under a graph's automorphism and represents the topology of the graph. In 1947[5], Wiener introduced the idea of the topological index while researching on alkane boiling ponts. In order to study quantitative structure-property relationships (QSPR) and quantitative structure-activity relationships (QSAR), topological indices are essential in chemical graph theory.

The connectivity index $\chi(G)$of a molecular graph G was introduced by randie [5] and is similar to the Zagreb group indices

2. Connectivity index of a graph

The connectivity index was introduced by Randie [5] as a bond additive quantity. It is similar to the Zagreb group indices. It was obtained after a search for the specific index to quantify the notion of molecular branching, unlike the indices $M_1(G)$ and $M_2(G)$ where G is a Molecular graph, which appeared in the topological formula for the π-electron energy of conjugated systems.

Let $G^+(Z_n, I_v)$, be an Involutory Additional Cayley Graph. The symbol $D(i)$ stands for the valency of a vertex i. The connectivity index may be viewed as a sum of bond weights given in terms of $[D(i)D(j)]^{-1/2}$.

Definition: The connectivity index $\chi(G)$of a molecular graph G was defined as

$$^{l}\chi(G) = \sum_{paths}[D(i)D(j) \dots D(l+1)]^{-1/2}$$

where $D(i), D(j), \dots, D(l+1)$ are valencies of vertices $i, j, \dots, l+1$ in the considered path l

The zero-order connectivity index was generalized as

$$^{0}\chi(G) = \sum_{i=1}^{N}[D(i)]^{-1/2}$$

where N is the total number of vertices in G or the total number of weighted paths p_0 of the length $l = 0$.

The first-order connectivity index was defined by

$$^{1}\chi(G) = \sum_{i,j=1}^{M}[D(i)D(j)]^{-1/2}$$

where M is the total number of edges in G or the total number of weighted paths $p_1$of the length $l = 1$.

The second-order connectivity index was generalized as

$$^{2}\chi(G) = \sum_{i,j,k=1}^{p_2}[D(i)D(j)D(k)]^{-1/2}$$

where p_2 is the number of all paths of the length two in G.

Theorem 2.1: For the involutory addition cayley graph $G^+(Z_n, I_v)$, if n is even and $n > 2$ then

(i) Zero connectivity index $^{0}\chi\ (G^+(Z_n, I_v)) = n(|I_v|)^{\frac{-1}{2}}$.

(ii) First connectivity index $^{1}\chi\ (G^+(Z_n, I_v))= 2n(|I_v|)^{-1}\ \ (if\ |e| = 2n)$.

$$= n(|I_v|)^{-1}\ \ (if\ |e| = n)$$

(iii) Second connectivity index $^{2}\chi\ (G^+(Z_n, I_v))= n(|I_v|-1)!\,(|I_v|)^{\frac{-3}{2}}\ \ (if\ |e| = 2n)$.

$$= n(|I_v|)^{\frac{-3}{2}}\ \ (if\ |e| = n).$$

Proof: Consider a graph $G^+(Z_n, I_v)$ with vertex set $Z_n = \{0,1,2,3, \dots, n-1\}$ where I_v denotes the set of involutory elements in Z_n.

Let n be even and $n > 2$

Then the degree of each vertex $x \epsilon z_n$ in $G^+(Z_n, I_v)$ is $|I_v|$.

Let e_{xy} represents the edge of $G^+(Z_n, I_v)$ which joining the vertices v_x, v_y of degrees d_x, d_y.

The total number of edges in the graph $G^+(Z_n, I_v)$ is n or $2n$ if n is even.

Then

(i)The zero connectivity index of $G^+(Z_n, I_v)$ when $|e_{xy}| = n\ (or) 2n$ is

$$^0\chi\ (G^+(Z_n, I_v)) = \sum_{i=1}^{n}[D(i)]^{\frac{-1}{2}}$$

$$= n(|I_v|)^{\frac{-1}{2}}$$

(ii) Case 1 : Suppose $|e_{xy}| = n$

The first connectivity index of $G^+(Z_n, I_v)$ is

$$^1\chi\ (G^+(Z_n, I_v)) = \sum_{i,j=1}^{M}[D(i)D(j)]^{\frac{-1}{2}}$$

$$= |e_{xy}|[D(i)D(j)]^{\frac{-1}{2}}$$

$$= n(|I_v||I_v|)^{\frac{-1}{2}}$$

$$= n(|I_v|^2)^{\frac{-1}{2}}$$

$$= n(|I_v|)^{-1}$$

Case 2 : Suppose $|e_{xy}| = 2n$

$$^1\chi\ (G^+(Z_n, I_v)) = \sum_{i,j=1}^{M}[D(i)D(j)]^{\frac{-1}{2}}$$

$$= |e_{xy}|[D(i)D(j)]^{\frac{-1}{2}}$$

$$= 2n(|I_v||I_v|)^{\frac{-1}{2}}$$

$$= 2n(|I_v|^2)^{\frac{-1}{2}}$$

$$= 2n(|I_v|)^{-1}$$

(iii) Case 1 : Suppose $|e_{xy}| = n$

The second connectivity index of $G^+(Z_n, I_v)$ having n number of paths of length 2 is

$$^2\chi\ (G^+(Z_n, I_v)) = \sum_{i,j,k=1}^{p_2}[D(i)D(j)D(k)]^{\frac{-1}{2}}$$

$$= |e_{xy}|[D(i)D(j)D(k)]^{\frac{-1}{2}}$$

$$= n(|I_v||I_v||I_v|)^{\frac{-1}{2}}$$

$$= n(|I_v|^3)^{\frac{-1}{2}}$$

$$= n(|I_v|)^{\frac{-3}{2}}$$

Case 2 : Suppose $|e_{xy}| = 2n$ and having $n(|I_v| - 1)!$number of paths of length 2

$$^2\chi\ (G^+(Z_n, I_v)) = \sum_{i,j,k=1}^{p_2}[D(i)D(j)D(k)]^{\frac{-1}{2}}$$

$$= |e_{xy}|[D(i)D(j)D(k)]^{\frac{-1}{2}}$$

$$= n(I_v - 1)!\ (|I_v||I_v||I_v|)^{\frac{-1}{2}}$$

$$= n(I_v - 1)!\ (|I_v|^3)^{\frac{-1}{2}}$$

$$= n(I_v - 1)!\ (|I_v|)^{\frac{-3}{2}}$$

Hence the Zero connectivity index $^0\chi\ (G^+(Z_n, I_v)) = n(|I_v|)^{\frac{-1}{2}}$, First connectivity index ${}^{1}\chi\,(G^{+}(Z_n, I_v))= 2n(|I_v|)^{-1}$ $(if\ |e| = 2n)$.

$= n(|I_v|)^{-1}$ $(if\ |e| = n)$

And Second connectivity index ${}^{2}\chi\,(G^{+}(Z_n, I_v))= n(|I_v| - 1)!\,(|I_v|)^{\frac{-3}{2}}$ $(if\ |e| = 2n)$.

$= n(|I_v|)^{\frac{-3}{2}}$ $(if\ |e| = n)$

Theorem 2.2: For the Involutory Addition Cayley Graph $G^{+}(Z_n, I_v)$, if n is odd and n is either prime or prime power, then

(i) Zero connectivity index ${}^{0}\chi\,(G^{+}(Z_n, I_v)) = (n - 2)(|I_v|)^{\frac{-1}{2}} + 2$.

(ii) First connectivity index ${}^{1}\chi\,(G^{+}(Z_n, I_v))= (n - 3)(|I_v|)^{-1} + (|I_v|)^{\frac{1}{2}}$.

(iii) Second connectivity index ${}^{2}\chi\,(G^{+}(Z_n, I_v))= (n - 4)(|I_v|)^{\frac{-3}{2}} + 1$.

Proof: Consider a graph $G^{+}(Z_n, I_v)$ with vertex set $z_n = \{0,1,2,3, \dots, n - 1\}$ where I_v denotes the set of involutory elements in z_n.

Let n be odd and n be either prime or prime power

Let e_{xy} represents the edge of $G^{+}(Z_n, I_v)$ which joining the vertices v_x, v_y of degrees $d_x d_y$.

The total number of edges in the graph $G^{+}(Z_n, I_v)$ is $n - 1$ in which two edges having their end vertex degrees $|I_v|$ and $|I_v| - 1$ and the remaining edges having both the end vertices are of degree $|I_v|$.Then

(i)The zero connectivity index of $G^{+}(Z_n, I_v)$ is

$$
\begin{aligned}
{}^{0}\chi\,(G^{+}(Z_n, I_v)) &= \textstyle\sum_{i=1}^{n}[D(i)]^{\frac{-1}{2}} \\
&= (n - 2)(|I_v|)^{\frac{-1}{2}} + 2(1)^{\frac{-1}{2}} \\
&= (n - 2)(|I_v|)^{\frac{-1}{2}} + 2 \\
&= (n - 2)(|I_v|)^{\frac{-1}{2}} + |I_v|
\end{aligned}
$$

(ii)The first connectivity index of $G^{+}(Z_n, I_v)$ is

$$
\begin{aligned}
{}^{1}\chi\,(G^{+}(Z_n, I_v)) &= \textstyle\sum_{i,j=1}^{M}[D(i)D(j)]^{\frac{-1}{2}} \\
&= (n - 3)(|I_v||I_v|)^{\frac{-1}{2}} + 2(|I_v|.\,1)^{\frac{-1}{2}} \\
&= (n - 3)(|I_v|^2)^{\frac{-1}{2}} + 2(|I_v|)^{\frac{-1}{2}} \\
&= (n - 3)(|I_v|)^{-1} + (|I_v|)(|I_v|)^{\frac{-1}{2}} \\
&= (n - 3)(|I_v|)^{-1} + (|I_v|)^{\frac{1}{2}}
\end{aligned}
$$

(iii) The second connectivity index of $G^{+}(Z_n, I_v)$ having $(n - 2)$ paths of length 2 is

$$
\begin{aligned}
{}^{2}\chi\,(G^{+}(Z_n, I_v)) &= \textstyle\sum_{i,j,k=1}^{p_2}[D(i)D(j)D(k)]^{\frac{-1}{2}} \\
&= (n - 4)(|I_v||I_v||I_v|)^{\frac{-1}{2}} + 2(|I_v||I_v|.\,1)^{\frac{-1}{2}} \\
&= (n - 4)(|I_v|^3)^{\frac{-1}{2}} + 2(|I_v|^2)^{\frac{-1}{2}} \\
&= (n - 4)(|I_v|)^{\frac{-3}{2}} + 2(|I_v|)^{-1} \\
&= (n - 4)(|I_v|)^{\frac{-3}{2}} + |I_v|(|I_v|)^{-1}
\end{aligned}
$$

$= (n-4)(|I_v|)^{\frac{-3}{2}} + 1$

Hence the Zero connectivity index ${}^0\chi\,(G^+(Z_n, I_v)) = (n-2)(|I_v|)^{\frac{-1}{2}} + 2$,

First connectivity index ${}^1\chi\,(G^+(Z_n, I_v))= (n-3)(|I_v|)^{-1} + (|I_v|)^{\frac{1}{2}}$.

And Second connectivity index ${}^2\chi\,(G^+(Z_n, I_v))= (n-4)(|I_v|)^{\frac{-3}{2}} + 1$.

Theorem 2.3: For the involutory addition cayley graph $G^+(Z_n, I_v)$, if n is neither odd prime nor odd prime power and $|I_v| = 4$, then

(i) Zero connectivity index ${}^0\chi\,(G^+(Z_n, I_v)) = |I_v|(|I_v|-1)^{\frac{-1}{2}} + n(|I_v|)^{\frac{-1}{2}} - (|I_v|)^{\frac{1}{2}}$.

(ii) First connectivity index ${}^1\chi\,(G^+(Z_n, I_v))= (|I_v|^2 - |I_v|)^{\frac{1}{2}} + \frac{(n-1)}{2} - 12(|I_v|)^{-1}$.

(iii) Second connectivity index ${}^2\chi\,(\,G^+(Z_n, I_v)\,) = (2|I_v|-1)(|I_v|-1)^{\frac{-1}{2}} + (|I_v|-2)(|I_v|)^{\frac{1}{2}}(|I_v|-1)^{-1} - (2|I_v|^2 - 2|I_v| - n(|I_v|-1)! + (|I_v|)!)(|I_v|)^{\frac{-3}{2}}$.

Proof: Consider a graph $G^+(Z_n, I_v)$ with vertex set $z_n = \{0,1,2,3, \dots, n-1\}$ where I_v denotes the set of involutory elements in z_n.

Let n be neither odd prime nor odd prime power

Let e_{xy} represents the edge of $G^+(Z_n, I_v)$ which joining the vertices v_x, v_y of degrees $d_x d_y$.

The total number of edges in the graph $G^+(Z_n, I_v)$ is$(\frac{n-1}{2})|I_v|$in which 12 edges having their end vertex degrees $|I_v|$ and $|I_v|-1$ and the remaining edges having both the end vertices are of degree $|I_v|$.Then

(i)The zero connectivity index of $G^+(Z_n, I_v)$ is

$${}^0\chi\,(G^+(Z_n, I_v))= \sum_{i=1}^{n}[D(i)]^{\frac{-1}{2}}$$

$$= |I_v|(|I_v|-1)^{\frac{-1}{2}} + (n-|I_v|)(|I_v|)^{\frac{-1}{2}}$$

$$= |I_v|(|I_v|-1)^{\frac{-1}{2}} + n(|I_v|)^{\frac{-1}{2}} - (|I_v|)(|I_v|)^{\frac{-1}{2}}$$

$$= |I_v|(|I_v|-1)^{\frac{-1}{2}} + n(|I_v|)^{\frac{-1}{2}} - (|I_v|)^{\frac{1}{2}}$$

(ii)The first connectivity index of $G^+(Z_n, I_v)$ is

$${}^1\chi\,(G^+(Z_n, I_v))= \sum_{i,j=1}^{M}[D(i)D(j)]^{\frac{-1}{2}}$$

$$= 12(|I_v|(|I_v|-1))^{\frac{-1}{2}} + [\left(\frac{n-1}{2}\right)|I_v| - 12](|I_v||I_v|)^{\frac{-1}{2}}$$

$$= 12(|I_v|^2 - |I_v|)^{\frac{-1}{2}} + [\left(\frac{n-1}{2}\right)|I_v| - 12](|I_v|^2)^{\frac{-1}{2}}$$

$$= 12(|I_v|^2 - |I_v|)^{\frac{-1}{2}} + [\left(\frac{n-1}{2}\right)|I_v| - 12](|I_v|)^{-1}$$

$$= 12(|I_v|^2 - |I_v|)^{\frac{-1}{2}} + \left(\frac{n-1}{2}\right)(|I_v|)(|I_v|)^{-1} - 12(|I_v|)^{-1}$$

$$= 12(|I_v|^2 - |I_v|)^{\frac{-1}{2}} + \left(\frac{n-1}{2}\right) - 12(|I_v|)^{-1}$$

(iii) Number of paths of length 2 is $= [|I_v|(|I_v|-1)] + (n-|I_v|)(|I_v|-1)!$

The second connectivity index of $G^+(Z_n, I_v)$ is

$${}^2\chi\,(G^+(Z_n, I_v))= \sum_{i,j,k=1}^{p_2}[D(i)D(j)D(k)]^{\frac{-1}{2}}$$

$= (|I_v|(2|I_v|-1))(|I_v|(|I_v|-1)|I_v|)^{\frac{-1}{2}} + (|I_v|-2)((|I_v|)[(|I_v|-1)(|I_v|)(|I_v|-1)]^{\frac{-1}{2}} + (|I_v|(|I_v|-1) + (n-|I_v|)(|I_v|-1)!) - (|I_v|(2|I_v|-1) + (|I_v|-2)|I_v|)(|I_v||I_v||I_v|)^{\frac{-1}{2}}$

$= (2|I_v|^2 - |I_v|)(|I_v|^2(|I_v|-1))^{\frac{-1}{2}} + (|I_v|^2 - 2|I_v|)[(|I_v|)(|I_v|-1)^2]^{\frac{-1}{2}} + ((|I_v|^2 - |I_v|) + (n - |I_v|)(|I_v|-1)!) - (2|I_v|^2 - |I_v| + |I_v|^2 - 2|I_v|))(|I_v|^3)^{\frac{-1}{2}}$

$= (2|I_v|^2 - |I_v|)\left(|I_v|^{-1}(|I_v|-1)^{\frac{-1}{2}}\right) + (|I_v|^2 - 2|I_v|)\left((|I_v|)^{\frac{-1}{2}}(|I_v|-1)^{-1}\right) + (-2|I_v|^2 + 2|I_v| + n(|I_v|-1)! - (|I_v|)!)(|I_v|)^{\frac{-3}{2}}$

$= (2|I_v|-1)\left((|I_v|-1)^{\frac{-1}{2}}\right) + (|I_v|-2)\left((|I_v|)^{\frac{1}{2}}(|I_v|-1)^{-1}\right) - (2|I_v|^2 - 2|I_v| - n(|I_v|-1)! + (|I_v|)!)(|I_v|)^{\frac{-3}{2}}$

Hence the Zero connectivity index ${}^0\chi\,(G^+(Z_n, I_v)) = |I_v|(|I_v|-1)^{\frac{-1}{2}} + n(|I_v|)^{\frac{-1}{2}} - (|I_v|)^{\frac{1}{2}}$,

First connectivity index ${}^1\chi\,(G^+(Z_n, I_v)) = (|I_v|^2 - |I_v|)^{\frac{1}{2}} + \frac{(n-1)}{2} - 12(|I_v|)^{-1}$

and Second connectivity index ${}^2\chi\,(G^+(Z_n, I_v)) = (2|I_v|-1)(|I_v|-1)^{\frac{-1}{2}} + (|I_v|-2)(|I_v|)^{\frac{1}{2}}(|I_v|-1)^{-1} - (2|I_v|^2 - 2|I_v| - n(|I_v|-1)! + (|I_v|)!)(|I_v|)^{\frac{-3}{2}}$.

3. Wiener number of involutory addition Cayley graph

The wiener number W was introduced in1947 by wiener [5] as the path number. The path number was defined as the number of bonds between all pairs of atoms in an acyclic molecule graph. In the original work by wiener [5], the wiener number was not, strictly speaking, formulated in graph-theoretical terms. It was defined with in the frame work of chemical graph theory later in 1971 by Hosoya [5], who pointed out that the wiener number can be obtained from the distance matrix of a graph.

In this section, some results related to wiener number of involutory addition Cayley graph were present.

Definition: "The wiener number $w = w(G)$ of a molecular graph G is defined as the half-sum of the elements of the distance matrix D. That means, $W = \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N} (D)_{kl}$, where$(D)_{kl}$ represents off-diagonal elements of D".

Theorem 3.1: For the involutory addition cayley graph $G^+(Z_m, I_q)$, if m is even and $m > 2$ then

(i) $w = (\frac{m}{2})^3$ if $|I_q| = 2$

(ii) $w = \frac{m^3+16m}{16}$ if $|I_q| = 4$

Proof: Consider a graph $G^+(Z_m, I_q)$, with vertex set $z_m = \{0,1,2,3,\dots,m-1\}$ where $|I_q|$ denotes the set of involutory elements in Z_m.

Let m be even and $m > 2$. Then the degree of each vertex $x \epsilon Z_m$ in $G^+(Z_m, I_q)$ is $|I_q|$.

Let e_{xy} represents the edge of $G^+(Z_m, I_q)$ which joining the vertices v_x, v_y of degrees d_x, d_y respectively.

The total number of edges in the graph $G^+(Z_m, I_q)$ is m or $2m$.

If m is even, $|I_q| = 2$. Then wiener number

(i) $W = \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N} (D)_{kl} = \sum_{k=1}^{N}\sum_{l=1}^{K} (D)_{kl}$

$\quad = m\left[1 + 2 + 3 + \cdots + \left(\frac{m-2}{2}\right)\right] + (\frac{m}{2})^2$

$= m\left[\left(1+\left(\frac{m-2}{2}\right)\right)\left(\frac{\left(\frac{m-2}{2}\right)}{2}\right)\right]+\frac{m^2}{4}$

$= m\left[\left(\frac{2+m-2}{2}\right)\left(\frac{m-2}{4}\right)\right]+\frac{m^2}{4}$

$= m\left[\frac{m(m-2)}{8}\right]+\frac{m^2}{4}$

$= m\left[\frac{m^2-2m}{8}\right]+\frac{m^2}{4}$

$= \frac{m^3-2m^2}{8}+\frac{m^2}{4}$

$= \frac{m^3-2m^2+2m^2}{8}=\frac{m^3}{8}=(\frac{m}{2})^3$

(ii) Wiener number when $\left|I_q\right| = 4$

$W = \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N} (D)_{kl} = \sum_{k=1}^{N}\sum_{l=1}^{K} (D)_{kl}$

$= \frac{1}{2}[m[2\left(1+2+3+\cdots+\frac{m}{4}\right)+2\left(1+2+\cdots+\left(\frac{m}{4}-1\right)\right)+2]]$

$= \frac{m}{2}[2\left(\left(1+\frac{m}{4}\right)\left(\frac{\left(\frac{m}{4}\right)}{2}\right)\right)+2\left[\left(1+\left(\frac{m}{4}-1\right)\right)\left(\frac{\frac{m}{4}-1-1}{2}\right)+\left(\frac{1+\frac{m}{4}-1}{2}\right)\right]+2]$

$= \frac{m}{2}[2\left(\left(\frac{4+m}{4}\right)\left(\frac{m}{8}\right)\right)+2\left[(\frac{m}{4})\left(\frac{\frac{m}{4}-2}{2}\right)+\left(\frac{m}{8}\right)\right]+2]$

$= \frac{m}{2}[2\left(\frac{4m+m^2}{32}\right)+2\left[(\frac{m}{4})\left(\frac{m-8}{8}\right)+\left(\frac{m}{8}\right)\right]+2]$

$= \frac{m}{2}[\left(\frac{4m+m^2}{16}\right)+2\left[\left(\frac{m^2-8m}{32}\right)+\left(\frac{m}{8}\right)\right]+2]$

$= \frac{m}{2}[\left(\frac{4m+m^2}{16}\right)+2\left[\left(\frac{m^2-8m+4m}{32}\right)\right]+2]$

$= \frac{m}{2}[\left(\frac{4m+m^2}{16}\right)+2\left(\frac{m^2-4m}{32}\right)+2]$

$= \frac{m}{2}[\left(\frac{4m+m^2}{16}\right)+\left(\frac{m^2-4m}{16}\right)+2]$

$= \frac{m}{2}\left(\frac{4m+m^2+m^2-4m+32}{16}\right)$

$= \frac{m}{2}\left(\frac{2m^2+32}{16}\right)=\frac{m}{2}\left(\frac{m^2+16}{8}\right)=\left(\frac{m^3+16m}{16}\right).$

Theorem 3.2: For the involutory addition cayley graph $G^+(Z_m, I_q)$, if m is odd and m is either prime or prime power, then $w = \frac{m(m^2-1)}{6}$.

Proof: Consider a graph $G^+(Z_m, I_q)$ with vertex set $Z_m = \{0,1,2,3,\ldots,m-1\}$ where I_q denotes the set of involutory elements in Z_m.

Let m be odd and m be either prime or prime power.

Let e_{xy} represents the edge of $G^+(Z_m, I_q)$ which joining the vertices v_x, v_y of degrees d_x, d_y rerspectively.

The total number of edges in the graph $G^+(Z_m, I_q)$ is $m-1$ in which two edges having their end vertex degrees $|I_q|$ and $\left|I_q\right|-1$ and the remaining edges having both the end vertices are of degree $|I_q|$.Then

$$W = \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N} (D)_{kl} = \sum_{k=1}^{N}\sum_{l=1}^{K} (D)_{kl}$$

$= 1 + (1+2) + (1+2+3) + \cdots + (1+2+\cdots+(m-1))$

$= \frac{(m-1)(m-1+1)(m-1+2)}{6}$

$= \frac{(m-1)m(m+1)}{6}$

$= \frac{(m^2-1)m}{6}$

$= \frac{m(m^2-1)}{6}$.

Theorem 3.3: For the involutory addition cayley graph $G^+(Z_{15}, I_q)$ the wiener number is $w = m^2 - 1$

Proof: Consider a graph $G^+(Z_m, I_q)$ with vertex set $z_m = \{0,1,2,3, \dots, m-1\}$ where I_q denotes the set of involutory elements in z_m.

Let m be 15

Let e_{xy} represents the edge of $G^+(Z_m, I_q)$ which joining the vertices v_x, v_y of degrees $d_x d_y$.

The total number of edges in the graph $G^+(Z_m, I_q)$ is$(\frac{m-1}{2})|I_q|$in which 12 edges having their end vertex degrees $|I_q|$ and $|I_q| - 1$ and the remaining edges having both the end vertices are of degree $|I_q|$.Then

$W = \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N} (D)_{kl} = \sum_{k=1}^{N}\sum_{l=1}^{K} (D)_{kl}$

$= 1(2m-2) + 2(3m-1) + 3(2m-6) + 4(m-6)$

$= 2m - 2 + 6m - 2 + 6m - 18 + 4m - 24$

$= 18m - 46$

$= (m+3)m - (3m+1)$

$= m^2 + 3m - 3m - 1$

$= m^2 - 1$

Theorem 3.4: For the involutory addition cayley graph $G^+(Z_m, I_q)$ the wiener number is $w = 29m - 37$

Proof: Consider a graph $G^+(Z_m, I_q)$ with vertex set $z_m = \{0,1,2,3, \dots, m-1\}$ where I_q denotes the set of involutory elements in z_m.

Let m be 21

Let e_{xy} represents the edge of $G^+(Z_m, I_q)$ which joining the vertices v_x, v_y of degrees $d_x d_y$.

The total number of edges in the graph $G^+(Z_m, I_q)$ is$(\frac{m-1}{2})|I_q|$ in which 12 edges having their end vertex degrees $|I_q|$ and $|I_q| - 1$ and the remaining edges having both the end vertices are of degree $|I_q|$.Then

$W = \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N} (D)_{kl} = \sum_{k=1}^{N}\sum_{l=1}^{K} (D)_{kl}$

$= 1(2m-2) + 2(3m+5) + 3(2m+6) + 4(m+6) + 5(m-3) + 6(m-12)$

$= 2m - 2 + 6m + 10 + 6m + 18 + 4m + 24 + 5m - 15 + 6m - 72$

$= 29m - 37$.

4. Schultz index of involutory addition Cayley graph

Schultz has introduced a topological index, named the Schultz index [6], for characterizing alkanes by an integer. This index denoted by $MTI(G)$ where G is a molecular graph, is based on the adjacency matrix $A(G)$, the distance matrix $D(G)$, and the valency matrix $V(G)$ of an alkane tree. The sum of elements $e_i(i = 1,2,\dots,N)$ of the row matrix $V[A + D](G)$ gives the Schultz index and has been shown to be a useful topological index for QSPR modelling.

Definition: "Schultz index denoted by $MTI(G)$ where G is a molecular graph, is based on the adjacency matrix $A(G)$, the distance matrix $D(G)$, and the valency matrix $V(G)$ of an alkane tree. The sum of elements $e_i(i = 1,2,\dots,N)$ of the row matrix $V[A + D](G)$ gives the Schultz index. That means, the Schultz index $MTI(G) = \sum_{i=1}^{N} e_i$, where N indicates number of elements and e_i indicates elements in the matrix $V[A + D](G)$".

In this section, we present some results related to Schultz index of involutory addition Cayley graph.

Theorem 4.1: For the involutory addition cayley graph $G^+(Z_m, I_q)$ if m is even and $m > 2$ then schultz index $MTI(G) = \frac{m^3+8m}{2}$ if $|I_q| = 2$.

Proof: Consider a graph $G^+(Z_m, I_q)$, with vertex set $Z_m = \{0,1,2,3,\dots,m-1\}$ where $|I_q|$ denotes the set of involutory elements in Z_m.

Let m be even and $m > 2$. Then the degree of each vertex $x \epsilon Z_m$ in $G^+(Z_m, I_q)$ is $|I_q|$.

Let e_{xy} represents the edge of $G^+(Z_m, I_q)$ which joining the vertices v_x, v_y of degrees d_x, d_y respectively.

The total number of edges in the graph $G^+(Z_m, I_q)$ is m.

If m is even, $|I_q| = 2$. Then Schultz index

$$
\begin{aligned}
MTI(G) &= V[A + D] \\
&= m[2\left(2 + 2 + 3 + 4 + \dots + \tfrac{m}{2}\right) + 2(2 + 2 + 3 + 4 + \dots + (\tfrac{m-2}{2}))] \\
&= m[2\left(2 + \left(2 + \tfrac{m}{2}\right)\left(\tfrac{\left(\frac{m}{2}-1\right)}{2}\right)\right) + 2(2 + \left(2 + \left(\tfrac{m-2}{2}\right)\right) + (\tfrac{2+\frac{m-2}{2}}{2}))] \\
&= m[2\left(2 + \left(\tfrac{4+m}{2}\right)\left(\tfrac{(m-2)}{4}\right)\right) + 2(2 + \left(\tfrac{4+m-2}{2}\right)(\tfrac{m-2-4}{4}) + (\tfrac{4+m-2}{4}))] \\
&= m[2\left(2 + \left(\tfrac{2m-8+m^2}{8}\right)\right) + 2(2 + \left(\tfrac{m+2}{2}\right)(\tfrac{m-6}{4}) + (\tfrac{m+2}{4}))] \\
&= m[2\left(\tfrac{16+2m-8+m^2}{8}\right) + 2(2 + (\tfrac{m^2-4m-12}{8}) + (\tfrac{m+2}{4}))] \\
&= m[2\left(\tfrac{m^2+2m+8}{8}\right) + 2(\tfrac{16+m^2-4m-12+2m+4}{8})] \\
&= m[\left(\tfrac{m^2+2m+8}{4}\right) + (\tfrac{m^2-2m+8}{4})] \\
&= m[\tfrac{m^2+2m+8+m^2-2m+8}{4}] \\
&= \tfrac{m(2m^2+16)}{4} \\
&= \tfrac{m(m^2+8)}{2} \\
&= \tfrac{m^3+8m}{2}
\end{aligned}
$$

Theorem 4.2: For the involutory addition cayley graph $G^+(Z_m, I_q)$, if m is even and $m > 2$ then schultz index $MTI(G) = \frac{m^3+48m}{2}$ if $|I_q| = 4$.

Proof: Consider a graph $G^+(Z_m, I_q)$, with vertex set $Z_m = \{0,1,2,3,\dots,m-1\}$ where $|I_q|$ denotes the set of involutory elements in Z_m.

Let m be even and $m > 2$. Then the degree of each vertex $x \epsilon Z_m$ in $G^+(Z_m, I_q)$ is $|I_q|$.

Let e_{xy} represents the edge of $G^+(Z_m, I_q)$ which joining the vertices v_x, v_y of degrees d_x, d_y respectively.

The total number of edges in the graph $G^+(Z_m, I_q)$ is $2m$.

If m is even, $|I_q| = 4$. Then Schultz index

$$MTI(G) = V[A + D]$$

$$= m[4(2\left(2+2+3+4+\cdots+\frac{m}{4}\right)+2\left(2+2+3+4+\cdots+\left(\frac{m}{4}-1\right)\right)+2)]$$

$$= 4m[2\left(2+\left(2+\frac{m}{4}\right)\left(\frac{\left(\frac{m}{4}-2\right)}{2}\right)+\frac{\left(2+\frac{m}{4}\right)}{2}\right)+2(2+\left(2+\left(\frac{m}{4}-1\right)\right)\left(\frac{\left(\frac{m}{4}-1-1\right)}{2}\right))+2]$$

$$= 4m[2\left(2+\left(\frac{8+m}{4}\right)\left(\frac{m-8}{8}\right)+\left(\frac{8+m}{8}\right)\right)+2\left(2+\left(\frac{8+m-4}{4}\right)\left(\frac{m-8}{8}\right)\right)+2]$$

$$= 4m[2\left(\frac{64+m^2-64+32+4m}{32}\right)+2\left(2+\left(\frac{m+4}{4}\right)\left(\frac{m-8}{8}\right)\right)+2]$$

$$= 4m[\left(\frac{m^2+4m+32}{16}\right)+2\left(\frac{64+m^2-4m-32}{32}\right)+2]$$

$$= 4m[\frac{m^2+4m+32}{16}+\frac{m^2-4m+32}{16}+2]$$

$$= 4m[\frac{m^2+4m+32+m^2-4m+32+32}{16}]$$

$$= 4m[\frac{2m^2+96}{16}]$$

$$= m[\frac{2m^2+96}{4}]$$

$$= m[\frac{m^2+48}{2}]$$

$$= \frac{m^3+48m}{2}.$$

Theorem 4.3: For the involutory addition cayley graph $G^+(Z_m, I_q)$, if m is odd and m is either prime or prime power, then schultz index

$$MTI(G) = \frac{2m^3-3m^2+13m-18}{3}.$$

Proof: Consider a graph $G^+(Z_m, I_q)$ with vertex set $Z_m = \{0,1,2,3,\dots,m-1\}$ where I_q denotes the set of involutory elements in Z_m.

Let m be odd and m be either prime or prime power.

Let e_{xy} represents the edge of $G^+(Z_m, I_q)$ which joining the vertices v_x, v_y of degrees d_x, d_y respectively.

The total number of edges in the graph $G^+(Z_m, I_q)$ is $m-1$ in which two edges having their end vertex degrees $|I_q|$ and $|I_q|-1$ and the remaining edges having both the end vertices are of degree $|I_q|$.Then

$$MTI = 2\left[2\left(1+(1+2)+(1+2+3)+\cdots+\left(1+2+3+\cdots+(m-1)\right)\right)\right]$$

$$- \qquad 2[1+2+3+\cdots+(m-1)]+4(m-2)+2$$

$$= 4\left[\frac{(m-1)m(m+1)}{6}\right]-2\left[\frac{m(m-1)}{2}\right]+4m-8+2$$

$= \frac{2m(m-1)(m+1)-3m(m-1)}{3} + 4m - 6$

$= \frac{m(m-1)(2(m+1)-3)}{3} + 4m - 6$

$= \frac{m(m-1)(2m+2-3)}{3} + 4m - 6$

$= \frac{(m^2-m)(2m-1)}{3} + 4m - 6$

$= \frac{2m^3-m^2-2m^2+m}{3} + 4m - 6$

$= \frac{2m^3-3m^2+m}{3} + 4m - 6$

$= \frac{2m^3-3m^2+m+12m-18}{3}$

$= \frac{2m^3-3m^2+13m-18}{3}$.

Theorem 4.4: For the involutory addition cayley graph $G^+(Z_{15}, I_q)$ the schultz index

$MTI(G) = \frac{4m^3+7m^2-8m-11}{8}$.

Proof: Consider a graph $G^+(Z_m, I_q)$ with vertex set $z_m = \{0,1,2,3,\dots,m-1\}$ where I_q denotes the set of involutory elements in z_m.

Let m be 15

Let e_{xy} represents the edge of $G^+(Z_m, I_q)$ which joining the vertices v_x, v_y of degrees $d_x d_y$.

The total number of edges in the graph $G^+(Z_m, I_q)$ is$(\frac{m-1}{2})|I_q|$in which 12 edges having their end vertex degrees $|I_q|$ and $|I_q| - 1$ and the remaining edges having both the end vertices are of degree $|I_q|$.Then

$MTI(G) = V[A + D]$

$= 4\big[2\big(1(2m-2) + 2(3m-1) + 3(2m-6) + 4(m-6)\big)\big] - 4[3(1+2+3+4) + 2(2)] + 4(4)(m - 4) + 9(4)$

$= 8(m^2-1) - 4\left[3\left(1 + 2 + \cdots + \left(\frac{m+1}{4}\right)\right) + 4\right] + 16(m-4) + 36$

$= 8m^2 - 8 - 4\left[3\left(1 + \left(\frac{m+1}{4}\right)\right)\left(\frac{m+1}{4}\Big/2\right) + 4\right] + 16m - 64 + 36$

$= 8m^2 + 16m - 36 - 4[3\left(\frac{m+5}{4}\right)\left(\frac{m+1}{8}\right) + 4]$

$= 8m^2 + 16m - 36 - 4[\frac{3(m^2+6m+5)}{32} + 4]$

$= 8m^2 + 16m - 36 - 4[\frac{3m^2+18m+15+128}{32}]$

$= \frac{64m^2+128m-288-3m^2-18m-143}{8}$

$= \frac{61m^2+110m-431}{8}$

$= \frac{(4m+1)m^2+(7m+5)m-(28m+11)}{8}$

$= \frac{4m^3+m^2+7m^2+5m-((m+13)m+11)}{8}$

$= \frac{4m^3+8m^2+5m-m^2-13m-11}{8}$

$= \frac{4m^3+7m^2-8m-11}{8}$

Theorem 4.5: For the involutory addition cayley graph $G^+(Z_{21}, I_q)$,the schultz index

$MTI(G) = \frac{4m^3-5m-11}{8}$.

Proof: Consider a graph $G^+(Z_m, I_q)$ with vertex set $z_m = \{0,1,2,3, \dots, m-1\}$ where I_q denotes the set of involutory elements in z_m.

Let m be 21

Let e_{xy} represents the edge of $G^+(Z_m, I_q)$ which joining the vertices v_x, v_y of degrees $d_x d_y$.

The total number of edges in the graph $G^+(Z_m, I_q)$ is$(\frac{m-1}{2})|I_q|$in which 12 edges having their end vertex degrees $|I_q|$ and $|I_q| - 1$ and the remaining edges having both the end vertices are of degree $|I_q|$.Then

$MTI(G) = V[A + D]$

$= 4[2(1(2m-2) + 2(3m+5) + 3(2m+6) + 4(m+6) + 5(m-3) + 6(m-12))] - 4\left[3\left(1 + 2 + \cdots + (\frac{m+3}{4})\right) + 2(2)\right] + 4(4)(m-4) + 9(4)$

$= 8[29m - 37] - 4\left[3\left(1 + \left(\frac{m+3}{4}\right)\right)\left(\frac{m+3}{4}\Big/2\right) + 4\right] + 16(m-4) + 36$

$= 232m - 296 - 4\left[3\left(\frac{m+7}{4}\right)\left(\frac{m+3}{8}\right) + 4\right] + 16m - 64 + 36$

$= 248m - 324 - 4[3\left(\frac{m^2+10m+21}{32}\right) + 4]$

$= 248m - 324 - 4[\frac{3m^2+30m+63+128}{32}]$

$= 248m - 324 - [\frac{3m^2+30m+191}{8}]$

$= \frac{1984m-2592-3m^2-30m-191}{8}$

$= \frac{-3m^2+1954m-2783}{8}$

$= \frac{-3m^2+(93m+1)m-(132m+11)}{8}$

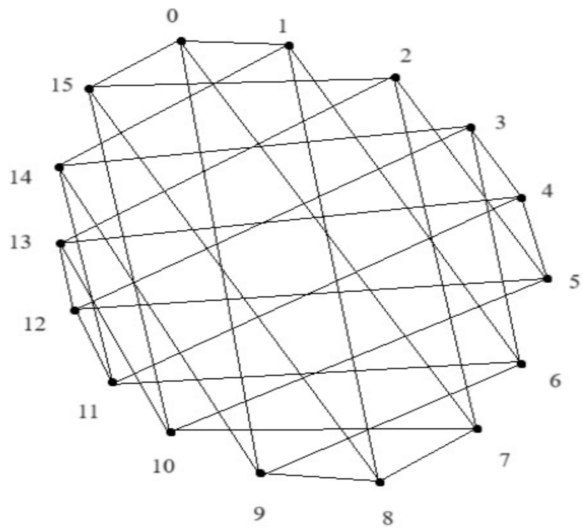
$= \frac{-3m^2+[(4m+9)m+1]m-[(6m+6)m+11]}{8}$

$= \frac{-3m^2+[4m^2+9m+1]m-[6m^2+6m+11]}{8}$

$= \frac{-3m^2+4m^3+9m^2+m-6m^2-6m-11}{8}$

$= \frac{4m^3-5m-11}{8}$.

Illustrations



$G^+(Z_{16}, I_q)$

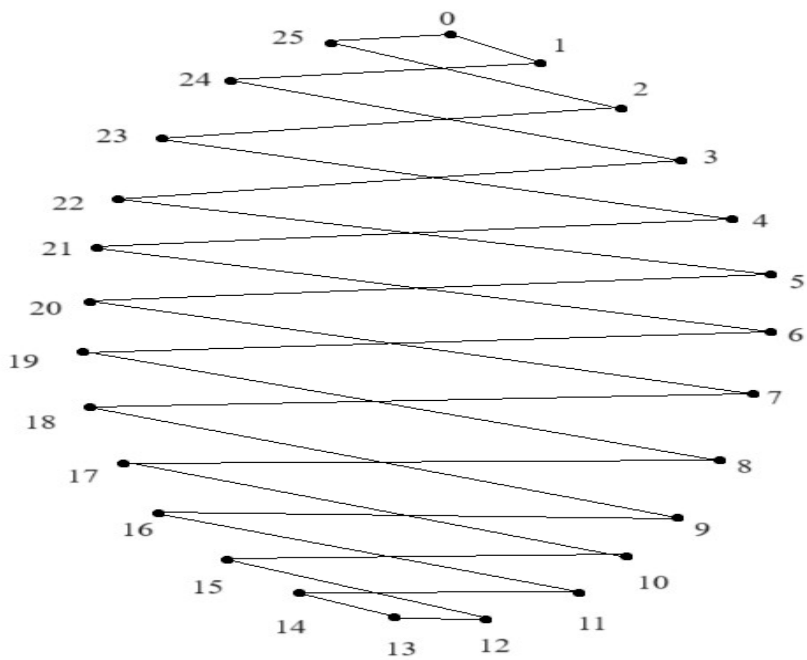
$^0\chi\ (G^+(Z_m, I_q)) = 8$

$^1\chi\ (G^+(Z_m, I_q)) = 8$

$^2\chi\ (G^+(Z_m, I_q)) = 12$

$W(G^+(Z_m, I_q)) = 272$

$MTI(G^+(Z_m, I_q)) = 2432$



$G^{+}(Z_{26}, I_q)$

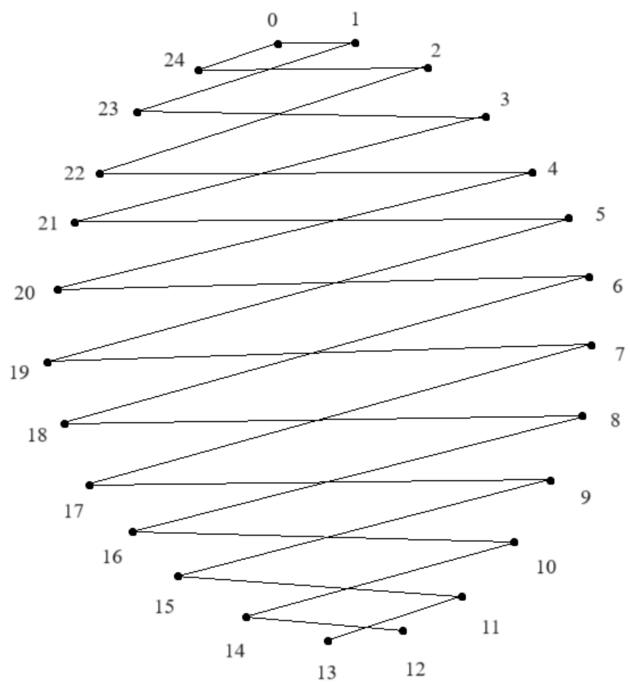
${}^{0}\chi\,(G^{+}(Z_m, I_q)) = 18.3848$

${}^{1}\chi\,(G^{+}(Z_m, I_q)) = 13$

${}^{2}\chi\,(G^{+}(Z_m, I_q)) = 9.1924$

$W(G^{+}(Z_m, I_q)) = 2197$

$MTI(G^{+}(Z_m, I_q)) = 8892$



$G^{+}(Z_{25}, I_q)$

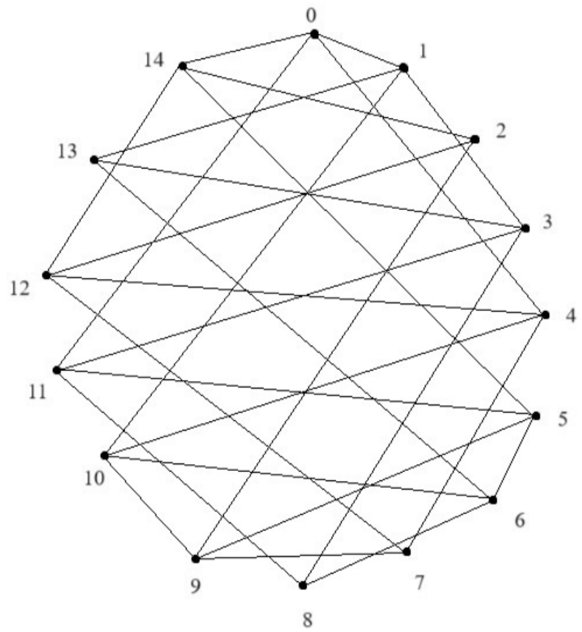
${}^{0}\chi\,(G^{+}(Z_m, I_q)) = 18.2634$

${}^{1}\chi\,(G^{+}(Z_m, I_q)) = 12.4142$

${}^{2}\chi\,(G^{+}(Z_m, I_q)) = 8.4246$

$W(G^{+}(Z_m, I_q)) = 2600$

$MTI(G^{+}(Z_m, I_q)) = 9894$



$G^{+}(Z_{15}, I_q)$

${}^{0}\chi\ (G^{+}(Z_m, I_q)) = 10$

${}^{1}\chi\ (G^{+}(Z_m, I_q)) = 10$

${}^{2}\chi\ (G^{+}(Z_m, I_q)) = 15$

$W(G^{+}(Z_m, I_q)) = 224$

$MTI(G^{+}(Z_m, I_q)) = 1868$

5. Conclusion

Using topological index, it is interesting to find the zero-connectivity index, first connectivity index and second connectivity index of involutory additional Cayley graphs and the authors have also studied this aspect.

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