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APPLICATION OF FIXED-POINT THEORY IN BIOLOGICAL SYSTEMS AND DISEASE SPREAD MODELLING

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Abstract:

Mathematical modelling of biological systems and infectious diseasetransmission has become an essential tool for understanding, predicting, and controlling epidemics. Fixed point theory provides a powerful mathematical framework for analyzing biological models by establishing the existence, uniqueness, and stability of equilibrium states. Many disease transmission models, including SIR, SEIR, epidemic network models, and reaction–diffusion systems, can be formulated as fixed point problems. Fixed point theorems such as the Banach Contraction Principle, Brouwer Fixed Point Theorem, Schauder Fixed Point Theorem, and Leray–Schauder theory are widely applied to investigate disease persistence, extinction, vaccination strategies, and population dynamics. This paper presents a survey of fixed point applications in biology, with particular emphasis on infectious disease spread, epidemic equilibrium, and modern mathematical approaches.

Keywords: *Fixed Point Theory, Mathematical Biology, Epidemic Models, Disease Spread, SIR Model, Stability Analysis, Nonlinear Differential Equations.*

1. Introduction

Fixed point theory is an important mathematical tool for analysing biological systems, especially those modelled by differential equations, difference equations, or dynamical systems. In biology, a fixed point (equilibrium point) represents a state in which the system remains unchanged over time if it is not disturbed. Biological systems are highly complex and often involve nonlinear interactions among individuals, populations, pathogens, and environmental factors. Infectious disease transmission is one of the most important examples where mathematical models are required to understand disease dynamics and design effective control strategies.

Traditional epidemic models such as the SIR model divide a population into compartments:

- **S**: Susceptible individual

- **I**: Infected individuals
- **R**: Recovered individuals

The evolution of disease spread is described by nonlinear differential equations. Because analytical solutions are often difficult or impossible to obtain, fixed point theory provides a mathematical approach to proving:

- Existence of epidemic equilibria,
- Uniqueness of solutions,
- Stability of disease-free states,
- Persistence of infection,
- Convergence of numerical simulations.

2. Fixed point concept in mathematical biology

A fixed point is a state that remains unchanged under a given transformation. For a biological system represented by $T : X \rightarrow X$, a fixed point satisfies $T(x)=x$.In disease modeling, a fixed point represents an equilibrium state where disease conditions remain constant.

Examples:

- Disease-free equilibrium
- Endemic equilibrium
- Stable population state
- Ecological balance

3. Epidemic models and fixed points

3.1 SIR (Susceptible–Infected–Recovered Disease) Model

The classical SIR model is given by: $\frac{dS}{dt} = -\beta S I$

$$\frac{dI}{dt} = \beta S I - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

where:S(t) = number of susceptible individuals at time t

I(t) = number of infected individuals at time t

R(t) = number of recovered individuals at time t

β = infection transmission rate

γ = recovery rate

The total population is:$N=S(t)+I(t)+R(t)$

An equilibrium point satisfies: $\frac{dS}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0$

These equilibrium solutions are fixed points of the epidemic system.

The SIR model was introduced by William Ogilvy Kermack **and** Anderson Gray McKendrick in 1927 and remains one of the most widely used mathematical frameworks for studying infectious disease dynamics. It has been extended into models such as SEIR (Susceptible–Exposed–Infected–Recovered)**,** SIRS (Susceptible–Infected–

Recovered–Susceptible), and network-based epidemic models.

4. Disease-free equilibrium

The disease-free equilibrium occurs when: $I=0$.

The population contains no infected individuals.

Fixed point analysis determines whether this state is stable.

If small infections introduced into the population disappear, the disease-free fixed point is stable.

If infections increase, the disease can spread.

5. Endemic Equilibrium: An **endemic equilibrium** is a steady-state solution of an epidemic model in which the disease persists in the population at a constant positive level. Unlike the disease-free equilibrium, the number of infected individuals remains positive over time.

An endemic equilibrium occurs when: $I>0$.

The disease remains permanently present in the population.

Fixed point methods help to determine:

- Whether endemic states exist,
- Whether infection persists,
- How parameters affect disease prevalence.

6. Banach fixed point theory in disease modeling

Many epidemic models can be transformed into integral equations: $T(u(t)) = u(t)$.

If T satisfies the contraction condition: $||T(u)-T(v)||\leq k||u-v||$, where $0<k<1$, Then Banach's theorem guarantees:

- The epidemic model has one and only one solution for the given initial conditions,
- Repeated numerical approximations converge to this solution,
- The predicted disease dynamics are mathematically well-defined and reproducible.

Applications

- Short-term epidemic prediction
- Viral infection models
- Population infection dynamics

7. Schauder fixed point theory in epidemic models

Many biological systems are nonlinear and do not satisfy contraction conditions.

Schauder's theorem is used when:

- The operator is continuous,
- The operator is compact,
- Solutions remain bounded.

Applications include:

- COVID-19 transmission models
- Vector-borne diseases
- HIV infection models
- Immunological systems

8. Fixed point applications in COVID-19 modeling

During the COVID-19 pandemic, mathematical models were used to estimate:

- Infection rates,
- Reproduction number,
- Hospital demand,
- Vaccination effects.

Fixed point methods were applied in: Nonlinear epidemic equations, fractional COVID-19 models, stochastic epidemic systems. The equilibrium points helped to analyze: disease elimination, endemic persistence, effects of intervention strategies.

9. Fractional differential models of disease spread

Fractional differential models have emerged as powerful tools for describing infectious disease dynamics because they incorporate **memory** effects and nonlocal behavior that are not captured by classical integer-order epidemic models. Unlike traditional SIR and SEIR models based on first-order derivatives, fractional models use derivatives of non-integer order to represent complex biological processes such as delayed infection, immunity loss, behavioral changes, and long-term effects of interventions. Fixed point theory provides an important mathematical framework for proving the existence, uniqueness, and stability of solutions of fractional epidemic models.

This paper presents the mathematical formulation of fractional disease models, their connection with fixed point theory, and their applications in COVID-19, HIV, influenza, malaria, and other infectious diseases.

A fractional epidemic model may be written as: $D^{\alpha} I(t) = F(I(t))$, where $0 < \alpha < 1$

Fixed point methods prove existence and uniqueness of solutions.

Applications

- COVID-19 dynamics
- HIV infection
- Tuberculosis
- Influenza

Advantages

- Captures biological problems, researchers can analyze disease equilibrium, stability, persistence, and control strategies. Applications range from classical SIR models to modern fractional differential equations, network epidemics, cancer modeling, and immunological systems. With the increasing complexity of biological interactions and emerging infectious diseases, fixed point methods will continue to be an important tool in mathematical biology and epidemiology memory.
- Represents delayed infection effects.
- Provides more realistic epidemic predictions.

10. Fixed point theory in vector-borne diseases

Fixed point theory is a fundamental mathematical tool used to study the long-term behavior of vector-borne disease models, such as Malaria, Dengue, Zika virus, West Nile virus, yellow fever etc. These diseases involve interactions between human hosts and vector populations (e.g., mosquitoes or ticks). A fixed point (equilibrium) represents a state where the disease dynamics no longer change with time.

Fixed point analysis determines:

- Disease persistence,
- Extinction conditions,
- Impact of vector control.

11. Reaction–Diffusion Models of Disease Spread

Disease spread is not only temporal but also spatial.

Reaction–diffusion equations describe: $\frac{\partial u}{\partial t} = D\nabla^2 u + F(u)$

where: D represents movement, F(u) represents biological interaction.

Applications

- Spatial epidemic spread
- Animal diseases
- Ecological invasion
- Tumor growth

Fixed point theory proves the existence of traveling wave solutions.

12. Fixed point theory in population dynamics

Biological populations can be modeled as: $T(x_n) = x_{n+1}$

A fixed point represents:

- Population equilibrium,
- Stable coexistence,
- Extinction.

Applications:

- Predator–prey systems
- Competition models
- Ecosystem stability

13. Disease control and vaccination strategies

Fixed point methods help to evaluate: vaccination coverage, treatment strategies, quarantine measures.

A control model can be written as: $T(x_n, u) = x_{n+1}$

where u represents intervention.

The fixed point determines the long-term disease state.

14. Network-based disease spread

Modern epidemics spread through social and contact networks.

A network epidemic model: $T(x_n) = x_{n+1}$ describes infection probability changes.

Fixed point analysis determines:

- Epidemic thresholds,

- Influential nodes,
- Effects of social distancing.

Applications:

- Social networks
- Air travel networks
- Hospital infection networks

15. Applications in cancer biology

Fixed point theory is widely used in **cancer biology** to analyse the long-term behaviour of mathematical models describing tumor growth, immune responses, angiogenesis, metastasis, and cancer treatment. In these models, a **fixed point (equilibrium point)** represents a biological state where the system no longer changes over time. Studying these equilibria helps researchers to determine whether a tumor will disappear, remain dormant, or continue growing.

Fixed point methods are also used in:

- Cancer cell population dynamics,
- Drug resistance models.

The equilibrium represents:

- Stable tumor size,
- Treatment response,
- Resistant population survival.

16. Applications in immunology

Fixed point theory is an important mathematical tool in immunology for studying the long-term behavior of immune system dynamics. Mathematical models of immune responses are commonly formulated using systems of ordinary differential equations (ODEs), partial differential equations (PDEs), or stochastic models. A fixed point (equilibrium point) represents a biological state in which immune cell populations, pathogens, cytokines, or antibodies remain constant over time. Analyzing these equilibria helps researchers understand infection clearance, immune homeostasis, autoimmune diseases, and responses to immunotherapy.

Fixed points represent

- Healthy immune balance,
- Chronic infection,
- Immune failure.

Applications

- HIV dynamics
- Viral persistence
- Autoimmune diseases

17. Numerical methods based on fixed points

Many computational biology methods use fixed point iteration: $T(x_n) = x_{n+1}$

Applications

- Epidemic simulation

- Parameter estimation
- Biological optimization
- Machine learning disease prediction models

18. Advantages of fixed point theory in disease modeling

- Provides rigorous mathematical analysis.
- Predicts long-term disease behavior.
- Determines epidemic stability.
- Helps evaluate intervention strategies.
- Supports computational simulation.
- Handles nonlinear biological interactions.

19. Challenges of fixed point theory in disease modeling

Fixed point theory is a powerful mathematical tool for proving the existence of solutions and equilibrium states in epidemiological models. However, its application to disease modeling also presents several theoretical and practical challenges.

- **Uncertainty in biological parameters:** Epidemiological models often involve uncertain or variable biological parameters, which can affect the applicability and accuracy of fixed point methods.
- **Difficulty in constructing the operator:** The first step in applying a fixed point theorem is to reformulate the epidemic model as an operator equation, . For simple SIR models, this is straightforward, but for models involving time delays, age structure, spatial diffusion, vaccination, and nonlinear incidence, constructing an appropriate operator becomes considerably more difficult.
- **Verifying the assumptions of fixed point theorems:** Each fixed point theorem requires specific hypotheses. For example, Schauder's Fixed Point Theorem requires that the operator is continuous, compact, and maps a closed, bounded, convex set into itself. Demonstrating these properties often involves advanced functional analysis, including boundedness estimates, equicontinuity, and the Arzelà–Ascoli theorem. For high-dimensional or strongly nonlinear epidemic models, these verifications can be technically demanding.
- **Non-uniqueness of solutions:** Many fixed point theorems, such as Schauder's Fixed Point Theorem, guarantee only the existence of a solution. They do **not** ensure that the solution is unique. Consequently, a disease model may admit multiple equilibria or multiple solution trajectories, making it difficult to determine which one is biologically relevant.
- **No information about stability:** A fixed point theorem establishes that an equilibrium exists but does not indicate whether it is stable. After proving existence, additional analyses—such as Jacobian eigenvalue analysis, Lyapunov functions, LaSalle's invariance principle, and the Routh–Hurwitz criteria—are required to determine whether the disease-free or endemic equilibrium is stable.
- **Limited biological interpretation:** The fixed point obtained is a mathematical object. Additional work is needed to verify that it satisfies biological constraints such as non-negative populations, bounded population sizes, realistic transmission dynamics, and epidemiologically meaningful parameter values. Without these checks, a mathematically valid fixed point may not correspond to a feasible biological state.

20. Future research directions

Future applications include:

- AI-based epidemic forecasting using fixed point algorithms.
- Real-time disease monitoring.
- Multi-pathogen models.
- Climate-driven disease prediction.
- Personalized medicine models.
- Digital epidemiology.

Conclusion

Fixed point theory provides a fundamental mathematical framework for understanding biological systems and infectious disease dynamics. By transforming epidemic models into fixed point problems, researchers can analyze disease equilibrium, stability, persistence, and control strategies. Applications range from classical SIR models to modern fractional differential equations, network epidemics, cancer modeling, and immunological systems. With the increasing complexity of biological interactions and emerging infectious diseases, fixed point methods will continue to be an important tool in mathematical biology and epidemiology.

References

1. Kermack, W. O., & McKendrick, A. G. (1927). A contribution to the mathematical theory of epidemics. *Proceedings of the Royal Society A, 115*, 700–721.
2. Hethcote, H. W. (2000). The mathematics of infectious diseases. *SIAM Review, 42*(4), 599–653.
3. Van den Driessche, P., & Watmough, J. (2002). Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Mathematical Biosciences, 180*, 29–48.
4. Banach, S. (1922). *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*. *Fundamenta Mathematicae, 3*, 133–181.
5. Granas, A., & Dugundji, J. (2003). *Fixed point theory*. Springer.
6. Smith, H. L. (1995). *Monotone dynamical systems: An introduction to the theory of competitive and cooperative systems*. American Mathematical Society.
7. Perko, L. (2001). *Differential equations and dynamical systems*. Springer.
8. Diekmann, O., Heesterbeek, J. A. P., & Britton, T. (2013). *Mathematical tools for understanding infectious disease dynamics*. Princeton University Press.