RESEARCH ARTICLE

ON IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract:

This paper is devoted to the study of existence and uniqueness of solutions for a class of implicit fractional differential equations involving the Caputo fractional derivative. The analysis is carried out by employing appropriate fixed-point techniques, including a modified contraction principle and the nonlinear alternative for contractions. Sufficient conditions ensuring the existence and uniqueness of solutions are established. Finally, an illustrative example is presented to demonstrate the applicability of the obtained theoretical results.

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1. Introduction:

It is well known that the behavior of many physical systems can be effectively described using various types of fractional differential equations. Consequently, the theory of fractional differential equations has attracted considerable attention due to its importance in modeling a wide range of phenomena. A detailed account of the fundamental theory of fractional calculus, fractional differential equations, and their applications can be found in the monographs [1-4]. We also refer to the works [5-8], where several interesting theoretical results concerning fractional differential equations have been investigated.

Recently, considerable attention has been devoted to implicit fractional differential equations. In particular, the existence, uniqueness, stability, and other qualitative properties of solutions for implicit fractional differential equations subject to initial and boundary conditions have been studied in [9-11]. These results were established by employing various fixed-point techniques and approximation methods.

Motivated by the works [9-11], in the present paper we investigate the existence and uniqueness of solutions to an initial value problem for a class of implicit fractional differential equations involving the Caputo fractional derivative, given in the following form:

$${}^{c}D^{\alpha}x(t) = f\left(t, x(t), {}^{c}D^{\alpha}x(t)\right), \quad t \in [0,b],\ b > 0,\ n-1 < \alpha \le n, \qquad (1.1)$$

$$x^{(k)}(0) = x_k \in \mathbb{R}^n, \quad k = 0,1,\dots,n-1, \qquad (1.2)$$

Where $f: [0,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear continuous function, $x: [0,b] \to \mathbb{R}^n$, and ${}^{c}D^{\alpha}$ denotes the Caputo fractional derivative of order α. Some special cases of problems (1.1)–(1.2) have been studied in [9-12] using various analytical approaches.

The remainder of this paper is organized as follows. In Section 2, we present some preliminary concepts, basic definitions from fractional calculus, and the auxiliary results required for our subsequent analysis. In Section 3, we establish the main existence and uniqueness results for problem (1.1)–(1.2). Finally, an illustrative example is provided to demonstrate the applicability of the obtained theoretical results.

2. Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space endowed with the norm $\|\cdot\|$. We denote by

$$B = C([0,b], \mathbb{R}^n)$$

the Banach space of all continuous functions from $[0,b]$ into $\mathbb{R}^n$, equipped with the supremum norm

$$\| x \|_B = \sup_{t \in [0,b]} \| x(t) \|.$$

Let $C([a,b], \mathbb{R})$ denote the space of all real-valued continuous functions defined on $[a,b]$, and let $C^n([a,b], \mathbb{R})$ denote the space of all real-valued functions on $[a,b]$ whose derivatives up to order n are continuous.

Now we recall some basic definitions and preliminary results from fractional calculus that will be used throughout this paper; see [6, 10, 12, 14] for further details.

Definition 2.1. Let $f \in C[0,b]$ and $\alpha \ge 0$. Then the Riemann–Liouville fractional integral of order α of a function f is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided the integral exists.

Definition 2.2. For a function $f \in C^n[0,b]$, the Caputo derivative of order α is defined as

$${}^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds, \quad n-1 < \alpha \le n.$$

Lemma 2.3. Let $f \in C^n[0,b]$ and $\alpha > 0$. Then

$$I^{\alpha}[{}^{c}D^{\alpha} f(t)] = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k+1)} t^k, \quad n-1 < \alpha \le n.$$

Lemma 2.4. Let $f(t) = t^{\beta}$, where $\beta \ge 0$, and let $n-1 < \alpha \le n$, $n \in \mathbb{N}$. Then

$${}^{c}D^{\alpha} f(t) = \begin{cases} 0, & \beta \in \{0,1,\dots,n-1\}, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, & \beta \in \mathbb{N},\ \beta \ge n \text{ or } \beta \notin \mathbb{N},\ \beta > n-1. \end{cases}$$

Definition 2.5. Let $\beta > 0$. Then the one-parameter Mittag–Leffler function of order $\beta > 0$ is defined by

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + 1)}.$$

Definition 2.6. If a function

$$f \in C([0,b] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n),$$

then the function $x \in B$ given by

$$x(t) = \sum_{k=0}^{n-1} \frac{x_k}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, x(s), {}^cD^\alpha x(s)\right) ds,$$

for $t \in [0,b]$, is the solution of implicit fractional differential equation (1.1)–(1.2).

To prove the existence result, we use the nonlinear alternative for contraction and modified version of contraction principle as mentioned below.

Theorem 2.7. [13] Non-linear alternative for contraction. Let U be a bounded open subset of Banach space E. Assume $0 \in U$ and $G: \overline{U} \to E$ is a contraction. Then either

(i) G has a unique fixed point in $\overline{U}$, or

(ii) there exist $\lambda \in (0,1)$ and $u \in \partial U$ such that

$$u = \lambda G u.$$

Theorem 2.8. [14] Modified version of contraction principle. Let X be a Banach space and let D be an operator which maps the elements of X into itself for which D^r is a contraction, where r is a positive integer. Then D has a unique fixed point.

3. Existence and uniqueness of solution

Firstly, we prove the existence and uniqueness results for (1.1)–(1.2) by using the non-linear alternative for contraction.

Theorem 3.1. *Let*

$$f \in C([0,b] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n).$$

Suppose $p_i(t) \in C([0,b], \mathbb{R}^+)$ *and* $L_i \in (0,1)$, $(i = 1,2)$, *such that*

(H1)

$$\| f(t,x,y) \| \leq p_1(t) \| x \| + L_1 \| y \|.$$

(H2)

$$\| f(t,x,y) - f(t,\tilde{x},\tilde{y}) \| \leq p_2(t) \| x - \tilde{x} \| + L_2 \| y - \tilde{y} \|,$$

where $t \in [0,b]$ *and* $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}^n$. *Then the initial-value problem* (1.1)–(1.2) *has a unique solution on* $[0,b]$, *provided*

$$\max\left\{\frac{p_1}{1-L_1}, \frac{p_2}{1-L_2}\right\} < \frac{\Gamma(\alpha+1)}{b^\alpha}, \qquad (3.1)$$

where

$$p_i = \max_{t \in [0,b]} p_i(t).$$

Proof. Consider, for any $\lambda \in (0,1)$, the family of problems

$$^{c}D^{\alpha}x(t) = \lambda f\big(t, x(t), {}^{c}D^{\alpha}x(t)\big), \quad t \in [0,b],\ b > 0,\ n-1 < \alpha \le n, \quad (3.2)$$

$$x^{(k)}(0) = \lambda x_k \in \mathbb{R}^n, \quad k = 0,1,\dots,n-1. \quad (3.3)$$

Then the solution of (3.2)–(3.3) is given by

$$x(t) = \sum_{k=0}^{n-1} \frac{\lambda x_k}{\Gamma(k+1)} t^k + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\big(s, x(s), {}^{c}D^{\alpha}x(s)\big)\, ds.$$

Using hypothesis (H1), we obtain

$$\| x(t) \| \le \sum_{k=0}^{n-1} \frac{\| x_k \|}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big\| f\big(s, x(s), {}^{c}D^{\alpha}x(s)\big)\big\|\, ds.$$

Hence

$$\| x(t) \| \le \sum_{k=0}^{n-1} \frac{\| x_k \|}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [p_1(s) \| x(s) \| + L_1 \| {}^{c}D^{\alpha}x(s) \|] ds.$$

Note that from (1.1), we have

$$\| {}^{c}D^{\alpha}x(t) \| = \big\| f\big(t, x(t), {}^{c}D^{\alpha}x(t)\big)\big\| \le p_1(t) \| x(t) \| + L_1 \| {}^{c}D^{\alpha}x(t) \|.$$

This gives

$$\| {}^{c}D^{\alpha}x(t) \| \le \frac{p_1(t)}{1-L_1} \| x(t) \|, \quad t \in [0,b].$$

Therefore,

$$\begin{aligned} \| x(t) \| &\le \sum_{k=0}^{n-1} \frac{\| x_k \|}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[p_1(s) \| x(s) \| + L_1 \frac{p_1(s)}{1-L_1} \| x(s) \|\right] ds \\ &\le \sum_{k=0}^{n-1} \frac{\| x_k \|}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{p_1(s)}{1-L_1} \| x(s) \|\ ds \\ &\le \sum_{k=0}^{n-1} \frac{\| x_k \|}{\Gamma(k+1)} t^k + \frac{p_1 \| x \|_B}{(1-L_1)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ ds \\ &= \sum_{k=0}^{n-1} \frac{\| x_k \|}{\Gamma(k+1)} t^k + \frac{p_1 \| x \|_B}{(1-L_1)\Gamma(\alpha)} \frac{t^\alpha}{\alpha}. \end{aligned}$$

Thus

$$\| x(t) \| \le \sum_{k=0}^{n-1} \frac{\| x_k \|}{\Gamma(k+1)} b^k + \frac{p_1 b^\alpha}{(1-L_1)\Gamma(\alpha+1)} \| x \|_B, \quad t \in [0,b].$$

Using condition (3.1), we can write

$$\| x \|_B \le \left(1 - \frac{p_1 b^\alpha}{(1-L_1)\Gamma(\alpha+1)}\right)^{-1} \sum_{k=0}^{n-1} \frac{\| x_k \|}{\Gamma(k+1)} b^k := \zeta.$$

We have proved that for any $\lambda \in (0,1)$ the solution of the family of problems is bounded and $\| x \|_B \le \zeta$.
Define

$$\overline{U} = \{x \in B : \| x \|_B \le \zeta\} = U \cup \partial U.$$

Define the operator $T: \overline{U} \to B$ by

$$(Tx)(t) = \sum_{k=0}^{n-1} \frac{x_k}{\Gamma(k+1)} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, x(s), {}^cD^\alpha x(s)\right) ds.$$

Then, for any $x, y \in \overline{U}$ and any $t \in [0, b]$,

$$\| (Tx)(t) - (Ty)(t) \| \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f\left(s, x(s), {}^cD^\alpha x(s)\right) - f\left(s, y(s), {}^cD^\alpha y(s)\right) \right\| ds.$$

Using (H2),

$$\| (Tx)(t) - (Ty)(t) \| \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[p_2(s) \| x(s) - y(s) \| + L_2 \| {}^cD^\alpha x(s) - {}^cD^\alpha y(s) \|\right] ds.$$

But, for any $t \in [0, b]$, from (1.1),

$$\begin{aligned} \| {}^cD^\alpha x(t) - {}^cD^\alpha y(t) \| &= \left\| f\left(t, x(t), {}^cD^\alpha x(t)\right) - f\left(t, y(t), {}^cD^\alpha y(t)\right) \right\| \\ &\le p_2(t) \| x(t) - y(t) \| + L_2 \| {}^cD^\alpha x(t) - {}^cD^\alpha y(t) \|. \end{aligned}$$

Therefore,

$$\| {}^cD^\alpha x(t) - {}^cD^\alpha y(t) \| \le \frac{p_2(t)}{1 - L_2} \| x(t) - y(t) \|.$$

Consequently,

$$\| (Tx)(t) - (Ty)(t) \| \le \frac{p_2}{1 - L_2} \frac{t^\alpha}{\Gamma(\alpha+1)} \| x - y \|_B \le \frac{p_2}{1 - L_2} \frac{b^\alpha}{\Gamma(\alpha+1)} \| x - y \|_B.$$

Thus

$$\| Tx - Ty \|_B \le \frac{p_2}{1 - L_2} \frac{b^\alpha}{\Gamma(\alpha+1)} \| x - y \|_B.$$

Condition (3.1) gives

$$\frac{p_2}{1 - L_2} \frac{b^\alpha}{\Gamma(\alpha+1)} < 1,$$

which shows that T is a contraction.

Now, if T has no fixed point in $\overline{U}$, then by the non-linear alternative for contraction, there exist $\lambda \in (0,1)$ and $u \in \partial U$ such that

$$u = \lambda T u.$$

This gives

$$\| u \| = \| \lambda T u \| = |\lambda| \, \| Tu \| < \| Tu \|,$$

which gives $\zeta < \zeta$, a contradiction. Hence T must have a unique fixed point in $\overline{U}$, which is the unique solution of (1.1)–(1.2).

This gives

$$\| u \| = \| \lambda T u \| = |\lambda| \, \| Tu \| < \| Tu \|,$$

which gives $\zeta < \zeta$, a contradiction. Hence T must have a unique fixed point in $\overline{U}$, which is the unique solution of (1.1)–(1.2).

Remark 3.1. With less hypothesis we can prove the existence and uniqueness of solution of (1.1)–(1.2) by using the modified version of contraction principle as follows.

Theorem 3.2. *If the function* $f \in C([0,b] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ *in (3.1) satisfies (H1), then (1.1)–(1.2) has a unique solution.*

Proof. Consider the operator T: B → B defined as in Theorem 3.1. We have already proved

$$\| (Tx)(t) - (Ty)(t) \| \le \frac{p_1}{1-L_1} \frac{t^\alpha}{\Gamma(\alpha+1)} \| x - y \|_B, \quad t \in [0,b].$$

For any t ∈ [0, b], define

$$x^* = Tx, \qquad y^* = Ty.$$

Then

$$\begin{aligned} \| (T^2x)(t) - (T^2y)(t) \| &= \| (Tx^*)(t) - (Ty^*)(t) \| \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \| f(s, x^*(s), {}^cD^\alpha x^*(s)) - f(s, y^*(s), {}^cD^\alpha y^*(s)) \| ds \\ &\le \frac{1}{\Gamma(\alpha)} \frac{p_1}{1-L_1} \int_0^t (t-s)^{\alpha-1} \| x^*(s) - y^*(s) \| \, ds \\ &= \frac{1}{\Gamma(\alpha)} \frac{p_1}{1-L_1} \int_0^t (t-s)^{\alpha-1} \| (Tx)(s) - (Ty)(s) \| \, ds \\ &\le \frac{p_1^2}{(1-L_1)^2} \frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)} \| x - y \|_B \int_0^t s^\alpha \, (t-s)^{\alpha-1} \, ds. \end{aligned}$$

Using the beta-function identity,

$$\int_0^t s^\alpha \, (t-s)^{\alpha-1} \, ds = t^{2\alpha} \frac{\Gamma(\alpha+1)\Gamma(\alpha)}{\Gamma(2\alpha+1)}.$$

Hence

$$\| (T^2x)(t) - (T^2y)(t) \| \le \frac{p_1^2}{(1-L_1)^2} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \| x - y \|_B.$$

Thus, in general, for any m ∈ ℕ, we find that

$$\| (T^mx)(t) - (T^my)(t) \| \le \frac{p_1^m}{(1-L_1)^m} \frac{t^{m\alpha}}{\Gamma(m\alpha+1)} \| x - y \|_B.$$

This gives

$$\| T^mx - T^my \|_B = \sup_{t\in[0,b]} \| (T^mx)(t) - (T^my)(t) \| \le \frac{((kb)^\alpha)^m}{\Gamma(m\alpha+1)} \| x - y \|_B,$$

where

$$k = \left(\frac{p_1}{1-L_1} \right)^{1/\alpha}.$$

Note that

$$\frac{((kb)^\alpha)^m}{\Gamma(m\alpha+1)}$$

is the mth term of the one-parameter Mittag–Leffler function

$$E_\alpha((kb)^\alpha) = \sum_{m=0}^\infty \frac{((kb)^\alpha)^m}{\Gamma(m\alpha+1)},$$

which is convergent. Therefore, by the fundamental theorem of Cauchy,

$$\frac{((kb)^{\alpha})^{m}}{\Gamma(m\alpha+1)} \to 0 \quad \text{as } m \to \infty.$$

Thus we can choose $m \in \mathbb{N}$ sufficiently large so that

$$\frac{((kb)^{\alpha})^{m}}{\Gamma(m\alpha+1)} < 1,$$

and hence T^m is a contraction. Therefore, by the modified version of contraction principle, T has a unique fixed point, which is the unique solution of (1.1)–(1.2).

4. Example

In this section, we give an example to illustrate the results.

Example 1. Consider the implicit differential equation of fractional order

$$^{c}D^{5/2}x(t) = \frac{e^{-t}}{2+t^2}\left(\frac{x(t)}{1-x(t)}\right) + \frac{1-\cos t}{8}\sin\left(\parallel {}^{c}D^{7/2}x(t) \parallel + \parallel x(t) \parallel\right), \quad t \in [0,2].$$

where $f: [0,2] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$f\left(t, x(t), {}^{c}D^{7/2}x(t)\right) = \frac{e^{-t}}{2+t^2}\left(\frac{x(t)}{1-x(t)}\right) + \frac{1-\cos t}{8}\sin\left(\parallel {}^{c}D^{7/2}x(t) \parallel + \parallel x(t) \parallel\right),$$

which is clearly continuous. Noting that

$$|\sin u - \sin v| = \left|2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)\right| \le |u-v|, \quad u, v \in \mathbb{R},$$

we have

$$\begin{aligned}
&\left\| f\left(t, x(t), {}^{c}D^{7/2}x(t)\right) - f\left(t, \tilde{x}(t), {}^{c}D^{7/2}\tilde{x}(t)\right) \right\| \\
&\le \frac{e^{-t}}{2+t^2} \parallel x(t) - \tilde{x}(t) \parallel + \frac{1}{4}\left|\sin\left(\parallel {}^{c}D^{7/2}x(t) \parallel + \parallel x(t) \parallel\right) - \sin\left(\parallel {}^{c}D^{7/2}\tilde{x}(t) \parallel + \parallel \tilde{x}(t) \parallel\right)\right| \\
&\le p_1(t) \parallel x(t) - \tilde{x}(t) \parallel + \frac{1}{4} \parallel {}^{c}D^{7/2}x(t) - {}^{c}D^{7/2}\tilde{x}(t) \parallel.
\end{aligned}$$

Hence, by Theorem 3.2, the above problem has a unique solution on [0,2].

Also, we observe that

$$\begin{aligned}
\left\| f\left(t, x(t), {}^{c}D^{7/2}x(t)\right) \right\| &\le \frac{e^{-t}}{2+t^2}\frac{\parallel x(t) \parallel}{\parallel 1-x(t) \parallel} + \frac{1}{4}\left|\sin\left(\parallel {}^{c}D^{7/2}x(t) \parallel + \parallel x(t) \parallel\right)\right| \\
&\le \frac{e^{-t}}{2+t^2} \parallel x(t) \parallel + \frac{1}{4}\left(\parallel {}^{c}D^{7/2}x(t) \parallel + \parallel x(t) \parallel\right) \\
&\le p_1(t) \parallel x(t) \parallel + \frac{1}{4} \parallel {}^{c}D^{7/2}x(t) \parallel.
\end{aligned}$$

Now, as

$$L_1 = L_2 = \frac{1}{4}, \qquad p_1 = p_2 = \frac{3}{4},$$

we find

$$\max\left\{\frac{p_1}{1-L_1}, \frac{p_2}{1-L_2}\right\} = \frac{3/4}{1-1/4} = 1,$$

and

$$\frac{\Gamma(\alpha+1)}{b^{\alpha}}=\frac{\Gamma\left(\frac{7}{2}+1\right)}{2^{7/2}}=1.0281.$$

In this case,

$$\max\left\{\frac{p_1}{1-L_1},\frac{p_2}{1-L_2}\right\}<\frac{\Gamma(\alpha+1)}{b^{\alpha}}.$$

Hence, by Theorem 3.1, the above problem has a unique solution on $[0,2]$.

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